MATH 100: MORE EXAMPLES FOR L'HÔPITAL'S RULE

1. Limits of the form 1^∞

We discuss three examples to show that limits of that form can have many possible limits.

(1) Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\sqrt{x}}$.

Solution 1: we take the logarithm. $\log\left(\left(1+\frac{1}{x}\right)^{\sqrt{x}}\right)=\sqrt{x}\log\left(1+\frac{1}{x}\right)$. Then

$$\lim_{x\to\infty}\sqrt{x}\log\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{x^{-1/2}}$$

is an indeterminate form ($\log\left(1+\frac{1}{x}\right)\xrightarrow[x\to\infty]{}\log(1+0)=0$ while $x^{-1/2}=\frac{1}{\sqrt{x}}\xrightarrow[x\to\infty]{}0$). We now apply l'Hôpital:

$$\begin{aligned}\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{x^{-1/2}} &= \lim_{x\to\infty}\frac{\left(-\frac{1}{x^2}\right)\left(\frac{1}{1+\frac{1}{x}}\right)}{-\frac{1}{2}x^{-3/2}}\\ &= \lim_{x\to\infty}\frac{2}{x^2\cdot x^{-3/2}}\cdot\frac{1}{1+\frac{1}{x}}\\ &= \lim_{x\to\infty}\frac{2}{\sqrt{x}\left(1+\frac{1}{x}\right)}=0\,.\end{aligned}$$

It follows that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\sqrt{x}}=e^0=1$.

Solution 2: we take the logarithm. $\log\left(\left(1+\frac{1}{x}\right)^{\sqrt{x}}\right)=\sqrt{x}\log\left(1+\frac{1}{x}\right)$. Then

$$\lim_{x\to\infty}\sqrt{x}\log\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{x^{-1/2}}$$

is an indeterminate form ($\log\left(1+\frac{1}{x}\right)\xrightarrow[x\to\infty]{}\log(1+0)=0$ while $x^{-1/2}=\frac{1}{\sqrt{x}}\xrightarrow[x\to\infty]{}0$). We now change variables to $u=\frac{1}{x}$ so that $u\to 0^+$ and apply l'Hôpital:

$$\begin{aligned}\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{x^{-1/2}} &= \lim_{u\to0^+}\frac{\log(1+u)}{u^{1/2}}\\ &= \lim_{u\to0^+}\frac{1/(1+u)}{\frac{1}{2}u^{-1/2}}=\lim_{u\to0^+}\frac{2u^{1/2}}{1+u}=\frac{2\cdot0^{1/2}}{1+0}=0\,.\end{aligned}$$

It follows that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\sqrt{x}}=e^0=1$.

(2) Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$.

Solution: we take the logarithm. $\log\left(\left(1+\frac{1}{x}\right)^{x}\right)=x\log\left(1+\frac{1}{x}\right)$. Then

$$\lim_{x\to\infty}x\log\left(1+\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{1/x}=\lim_{u\to0^+}\frac{\log(1+u)}{u}$$

is an indeterminate form ($\log(1) = 0$). By l'Hôpital,

$$\lim_{u \to 0^+} \frac{\log(1+u)}{u} = \lim_{u \to 0^+} \frac{1/(1+u)}{1} = \frac{1}{1+0} = 1\,.$$

It follows that $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{\sqrt{x}} = e^0 = 1$.

(3) Find $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x^2}$.

Solution: we take the logarithm. $\log\left(\left(1 + \frac{1}{x}\right)^{x^2}\right) = x^2 \log\left(1 + \frac{1}{x}\right)$. Then

$$\lim_{x\to\infty} x^2 \log\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\log\left(1 + \frac{1}{x}\right)}{1/x^2} = \lim_{u \to 0^+} \frac{\log(1+u)}{u^2}$$

is an indeterminate form ($\log(1) = 0$). By l'Hôpital,

$$\lim_{u \to 0^+} \frac{\log(1+u)}{u^2} = \lim_{u \to 0^+} \frac{1/(1+u)}{2u}\,.$$

Now $\lim_{u\to 0} \frac{1}{1+u} = 0$ while $\lim_{u\to 0^+} \frac{1}{u} = \infty$ so

$$\lim_{x\to\infty} \log\left(\left(1 + \frac{1}{x}\right)^{x^2}\right) = \lim_{u \to 0^+} \frac{1/(1+u)}{2u} = \infty$$

and hence

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x^2} = \lim_{x\to\infty} \exp\left(\log\left(\left(1 + \frac{1}{x}\right)^{x^2}\right)\right) = \infty\,.$$