Math 100: Summary of Limits

September 10, 2014

- Limit laws: limits respect arithmetic operations and common functions, as long as everything is *well-defined*.

Ideas for evaluating $\lim_{x\to a} f(x)$.

1. Is the expression *well-behaved* at a? If so, we can evaluate it.
2. Do we need to examine the limits from left and right separately?
   - If the one-sided limits are different, the two-sided "limit as $x \to a$" does not exist.
3. Does the function *blow up*? (e.g. $\frac{\sqrt{1+x}}{x}$ near $x = 0$?)
   - If the magnitude is going to ∞, check the *sign* at both sides (here, the limit as $x \to 0+$ is $+\infty$ but as $x \to 0-$ is $-\infty$)
4. Does the function oscillate between different values, so there is no limit? (example: $\sin\left(\frac{\pi}{x}\right)$ near zero).
5. The expression looks like $\frac{0}{0}$, many things can happen.
   (a) Can we apply a trick? We saw two in class:
      i. Cancel factors in numerator and denominator. For example, if $x \neq 3$ then
      $$\frac{x-3}{x^2-x-6} = \frac{1}{x+2}\,.$$
      ii. Rationalize roots. For example,
      $$\begin{aligned}\frac{\sqrt{\cos x + x} - \sqrt{\cos x - x}}{x} &= \frac{\sqrt{\cos x + x} - \sqrt{\cos x - x}}{x} \cdot \frac{\sqrt{\cos x + x} + \sqrt{\cos x - x}}{\sqrt{\cos x + x} + \sqrt{\cos x - x}} \\ &= \frac{\left(\sqrt{\cos x + x}\right)^2 - \left(\sqrt{\cos x - x}\right)^2}{x\left(\sqrt{\cos x + x} + \sqrt{\cos x - x}\right)} \\ &= \frac{(\cos x + x) - (\cos x - x)}{x\left(\sqrt{\cos x + x} + \sqrt{\cos x - x}\right)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{2x}{x}\cdot\frac{1}{\sqrt{\cos x + x}+\sqrt{\cos x - x}} \\
&= \frac{2}{\sqrt{\cos x + x}+\sqrt{\cos x - x}}
\end{aligned}
$$

thus

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{\cos x + x}-\sqrt{\cos x - x}}{x} &= \lim_{x\to 0}\frac{2}{\sqrt{\cos x + x}+\sqrt{\cos x - x}} \\
&= \frac{2}{\sqrt{\cos 0 + 0}+\sqrt{\cos 0 - 0}} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1\,.
\end{aligned}
$$

6. Can we *sandwich* the function between others whose limits we can calculate?

$$-1 \le \sin\frac{1}{x} \le 1$$

so

$$-x^2 \le x^2\sin\frac{1}{x} \le x^2$$

and $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}(x^2) = 0$ so

$$\lim_{x\to 0} x^2\sin\frac{1}{x} = 0\,.$$

More tips:

1. Don't forget the "limit" symbol. Compare the following:

$$\lim_{x\to 1}\frac{e^x\,(x-1)}{x^2+x-2} = \frac{e^x\,(x-1)}{(x-1)(x+2)} = \frac{e^x}{x+2} = \frac{e}{3} \quad \text{(wrong)}$$

and

$$\lim_{x\to 1}\frac{e^x\,(x-1)}{x^2+x-2} = \lim_{x\to 1}\frac{e^x\,(x-1)}{(x-1)(x+2)} = \lim_{x\to 1}\frac{e^x}{x+2} = \frac{\lim_{x\to 1} e^x}{\lim_{x\to 1}(x+2)} = \frac{e}{3} \quad \text{(correct)}$$

$$\lim_{x\to 1}\frac{e^x\,(x-1)}{x^2+x-2} = \lim_{x\to 1}\frac{e^x}{x+2} = \frac{e}{3} \quad \text{(succint but correct)}$$

(the problem with the first attempt is that the sign = means "really equal" not "the next step in the calculation is").