Math 539: Problem Set 1 (due 29/1/2014)

1. (The standard divisor bound)
   (a) Let $f(n)$ be multiplicative, and suppose that $f \to 0$ along prime powers – that is, for every $\varepsilon > 0$ there is N such that if $p^m > N$ then $|f(p^m)| \le \varepsilon$. Show that $\lim_{n\to\infty} f(n) = 0$.
   (b) Show that for all $\varepsilon > 0$, $d(n) = O(n^\varepsilon)$.

2. Establish the following identities in the ring of formal Dirichlet series
   (a) Let $d_k(n) = \sum_{\prod_{i=1}^k a_i = n} 1$ be the generalized divisor functions, counting factorizations of n into k parts (so $d_2(n) = d(n)$ is the usual divisor function). Show $\sum_n d_k(n) n^{-s} = (\zeta(s))^k$ and that $d_k * d_l = d_{k+l}$.
   (b) Define $d_{1/2}(n)$. Calculate $d_{1/2}(p)$, $d_{1/2}(12)$.
   (c) Let $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ be the generalized sum-of-divisors function. Show that $\sum_{n=1}^\infty \sigma_\alpha(n) n^{-s} = \zeta(s)\zeta(s-\alpha)$.
   (d) Show that $\sum_{n\ge1} d(n^2) n^{-s} = \frac{\zeta(s)^3}{\zeta(2s)}$.

3. Let $D_\varphi(s) = \sum_{n\ge1} \varphi(n) n^{-s}$.
   (a) Represent the series in terms of $\zeta(s)$ formally.
   (b) Show the series converges absolutely for $\Re(s) > 2$.
   (c) Show that the series failes to converge at $s = 2$.

4. Recall the function $\mathrm{Li}(x) = \int_2^x \frac{\mathrm{d}t}{\log t}$.
   (a) ("Asymptotic expansion") Show that for fixed K, $\mathrm{Li}(x) = \sum_{k=1}^K (k-1)! \frac{x}{\log^k x} + O_K\left(\frac{x}{\log^{K+1} x}\right)$.
   (b) (Asymptotic expansions are not series expansions) Show that $\sum_{k=1}^\infty (k-1)! \frac{x}{\log^k x}$ diverges.
   (c) Use summation by parts to estimate $\pi(x) = \sum_{p\le x} 1$ using the known asymptotics for $\sum_{p\le x} \frac{1}{p}$. Can you show $\pi(x) \ll \mathrm{Li}(x)$? $\pi(x) \gg \mathrm{Li}(x)$? That $\frac{\pi(x)}{\mathrm{Li}(x)} = 1 + o(1)$?

   NOTATION $f = \Theta(g)$ means $f = O(g)$ and $g = O(f)$, that is $0 \le \frac{1}{C} f(x) \le g(x) \le C f(x)$ for all large enough x.
   (d) Deduce $\pi(x) = \Theta(\mathrm{Li}(x))$ from Chebychev's estimate $\psi(x) = \Theta(x)$.

5. (Chebychev's lower bound) Let $\theta(x) = \sum_{p\le x} \log p$. We will find an explicit $c > 0$ such that $\theta(x) \ge cx$ for $x \ge 2$.
   (a) Let $v_p(n)$ denote the number of times p divides n. Show that $v_p(n!) = \sum_{k=1}^\infty \left\lfloor \frac{n}{p^k} \right\rfloor$.
   (b) Show that if $n < p \le 2n$ then $v_p\left(\binom{2n}{n}\right) = 1$.
   (c) (main saving) Show that if $\frac{2}{3}n < p \le n$ then $v_p\left(\binom{2n}{n}\right) = 0$ unless $n = p = 2$.
   (d) Show that if $\sqrt{2n} < p \le n$ then $v_p\left(\binom{2n}{n}\right) \le 1$.
   (e) Show that $v_p\left(\binom{2n}{n}\right) \le \log_p 2n$.
   (f) Show that $\log\binom{2n}{n} - (\theta(2n) - \theta(n)) \le \theta\left(\frac{2n}{3}\right) + 2\sqrt{2n}\log(2n)$.
   (g) Find a constant $c > 0$ such that $\theta(x) \ge cx$ for $2 \le x \le 4$ and such that if $\theta(x) \ge cx$ for all $2 \le x \le X$ then $\theta(x) \ge cx$ for $X < x \le 2X$.

6. Notation: $f(x) = o(g(x))$ ("little oh") if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0$.
   (a) Show $\prod_p \left(1 - \frac{1}{p}\right) e^{\frac{1}{p}}$ converges.
   (b) Show that $\prod_{p\le z} \left(1 - \frac{1}{p}\right) = \frac{C(1+o(1))}{\log z}$.

7. Let $\sigma > 0$
   (a) Show that $\prod_{p\le x}(1 + p^{-\sigma}) \le \exp\left(O(x^{1-\sigma}/\log x)\right)$.
   (b) Let $a_p \in \mathbb{C}$ satisfy $\left|a_p\right| \le p^{-\sigma}$. Show that $f(n) = \prod_{p|n}(1 + a_p) \le \exp\left(O\left((\log^{1-\sigma} n)(\log\log n)^{-1}\right)\right)$.
   (c) Show that $\sum_{n\le x} f(n) = cx + O(x^{1-\sigma})$ where $c = \prod_p \left(1 + \frac{a_p}{p}\right)$.

8. Let $\mathcal{A}_n$ denote a set of representative for the isomorphism classes of abelian groups of order n, $A_n = \#\mathcal{A}_n$ the number of isomorphism classes.
   (a) Show that $\sum_{n\ge1} A_n n^{-s} = \prod_{k=1}^{\infty} \zeta(ks)$ in the ring of formal Dirichlet series.
   (b) Show that $\sum_{n\le x} A_n = cx + O\left(x^{1/2}\right)$ where $c = \prod_{k=2}^{\infty} \zeta(k)$.

9. Let $A \subset \mathbb{N}$. We define its lower and upper *natural densities* by
$$\underline{d} = \liminf_{N\to\infty} \frac{1}{N} \sum_{n\le N} A(n)\,, \qquad \bar{d} = \limsup_{N\to\infty} \frac{1}{N} \sum_{n\le N} A(n)\,.$$
   If the two are equal the limit exists and we call it the *natural density* of A. Similarly, the lower and upper *logarithmic densities* are
$$\underline{\delta} = \liminf_{N\to\infty} \frac{1}{\log N} \sum_{n\le N} \frac{A(n)}{n}\,, \qquad \bar{\delta} = \limsup_{N\to\infty} \frac{1}{\log N} \sum_{n\le N} \frac{A(n)}{n}\,,$$
   and a set has a *logarithmic density* if it lower and upper densities agree.
   (a) Show that $0 \le \underline{d} \le \underline{\delta} \le \bar{\delta} \le \bar{d} \le 1$ for all A. Conclude that if A has natural density it has logarithmic density and the two are equal.
   (b) Let A be the set of integers whose most significant digit is 4. Compute the lower and upper natural and logarithmic densities of A. Does it have natural density? Logarithmic density?

Hint for 4(a): Repeatedly integrate by parts, and for the error estimate effectuate the mantra "log is a constant function" by breaking the interval of integration in two.

Supplementary problems

A. Give the ring of formal Dirichlet series the *ultrametric* topology. In other words, say that $D_n(s) \to D(s)$ if each coefficient eventually stabilizes. This allows us to define products of Dirichlet series.

   (a) Let $G(T) = \sum_{n=0}^{\infty} a_n T^n \in \mathbb{C}[[T]]$ be a formal power series, and let f be an arithmetic function with $f(1) = 0$. Show that $G(D_f) = \sum_{n=0}^{\infty} a_n \left(D_f(s)\right)^n$ is a convergent series in the ring of formal Dirichlet series.

   (b) Let $D(s) = \sum_{n\geq 1} b_n n^{-s}$ be a formal Dirichlet series with $b_1 = 1$. Realize $\log D(s)$ as a formal Dirichlet series without constant term, and show that $\exp\log D(s) = D(s)$.

   (c) If f is multiplicative then $\sum_{n\geq 1} f(n)n^{-s} = \prod_p \left(\sum_{m=0}^{\infty} f(p^m)p^{-s}\right)$, in the sense that the product on the right converges in the ultrametric topology.

   (d) (The complex topology is different) Give a multiplicative function f and a number s such that $\prod_p \left(1 + \left|\sum_{m=1}^{\infty} f(p^m)p^{-s}\right|\right)$ converges but $\sum_{n\geq 1} f(n)n^{-s}$ does not.

B. For each $z \in \mathbb{C}$ define a multiplicative function $d_z(n)$ giving a natural generalization of d_k, and satisfying $d_z * d_w = d_{z+w}$. Evaluate $d_z(p)$, $d_z(p^k)$, $d_z(360)$.

[Note for the future: problem on sum-free sets]