Math 412: Problem set 8, due 17/3/2014

Practice: Norms

P1. Call two norms $\|\cdot\|_1, \|\cdot\|_2$ on V *equivalent* if there are constants c, C such that for all $\underline{v} \in V$,

$$c\|\underline{v}\|_1 \leq \|\underline{v}\|_2 \leq C\|\underline{v}\|_1 .$$

(a) Show that this is an equivalence relation.

(b) Suppose the two norms are equivalent and that $\lim_{n\to\infty}\|\underline{v}_n\|_1 = 0$ (that is, that $\underline{v}_n \xrightarrow[n\to\infty]{\|\cdot\|_1} \underline{0}$). Show that $\lim_{n\to\infty}\|\underline{v}_n\|_2 = 0$ (that is, that $\underline{v}_n \xrightarrow[n\to\infty]{\|\cdot\|_2} \underline{0}$).

(**c) Show the converse of (b) also holds. In other words, two norms are equivalent iff they determine the same notion of convergence.

P2. Constructions

(a) Let $\{(V_i, \|\cdot\|_i)\}_{i=1}^n$ be normed spaces, and let $1 \leq p \leq \infty$. For $\underline{v} = (\underline{v}_i) \in \bigoplus_{i=1}^n V_i$ define

$$\|\underline{v}\| = \left(\sum_{i=1}^n \|\underline{v}_i\|_i^p\right)^{1/p} .$$

Show that this defines a norm on $\bigoplus_{i=1}^n V_i$.

DEF This operation is called the L^p*-sum* of the normed spaces.

DEF Let $(V, \|\cdot\|)$ be a normed space, and let $W \subset V$ be a subspace. For $\underline{v} + W \in V/W$ set $\|\underline{v} + W\|_{V/W} = \inf\{\|\underline{v} + \underline{w}\| : \underline{w} \in W\}$. Show

(b) Show that $\|\cdot\|_{V/W}$ is 1-homogenous and satisfies the triangle inequality (it's not always a norm because it can be zero for non-zero vectors).

Norms

1. Let $f(x) = x^2$ on $[-1,1]$.

(a) For $1 \leq p < \infty$. Calculate $\|f\|_{L^p} = \left(\int_0^1 |f(x)|^p \,\mathrm{d}x\right)^{1/p}$.

(b) Calculate $\|f\|_{L^\infty} = \sup\{|f(x)| : -1 \leq x \leq 1\}$. Check that $\lim_{p\to\infty}\|f\|_{L^p} = \|f\|_\infty$.

(c) Calculate $\|f\|_{H^2} = \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}$.

SUPP Show that the H^2 norm is equivalent to the norm $\left(\|f\|_{L^2}^2 + \|f''\|_{L^2}^2\right)^{1/2}$.

2. Let $A \in M_n(\mathbb{R})$.

(a) Show $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$ (hint: we basically did this in class).

(b) Show that $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$.

RMK See below on *duality*.

3. The *spectral radius* of $A \in M_n(\mathbb{C})$ is the magnitude of its largest eigenvalue: $\rho(A) = \max\{|\lambda| \mid \lambda \in \mathrm{Spec}(A)\}$.

(a) Show that for any norm $\|\cdot\|$ on F^n and any $A \in M_n(F)$, $\rho(A) \leq \|A\|$.

(b) Suppose that A is diagonable. Show that there is a norm on F^n such that $\|A\| = \rho(A)$.

(*c) Show that if A is Hermitian then $\|A\|_2 = \rho(A)$.

(d) Show that if A,B are similar, and $\|\cdot\|$ is any norm in $\mathbb{C}^n$, then $\lim_{n\to\infty}\|A^n\|^{1/n} = \lim_{n\to\infty}\|B^n\|^{1/n}$ (in the sense that, if one limit exists, then so does the other, and they are equal).

(**e) Show that for any norm on $\mathbb{C}^n$ and any $A \in M_n(\mathbb{C})$, we have $\lim_{n\to\infty}\|A^n\|^{1/n} = \rho(A)$.

4. The *Hilbert–Schmidt* norm on $M_n(\mathbb{C})$ is $\|A\|_{\mathrm{HS}} = \left(\sum_{i,j=1}^n \left|a_{ij}\right|^2\right)^{1/2}$.
   - Show that $\|A\|_{\mathrm{HS}} = \left(\mathrm{Tr}(A^\dagger A)\right)^{1/2}$.
   (a) Show that this is, indeed, a norm.
   (b) Show that $\|A\|_2 \le \|A\|_{\mathrm{HS}}$.

Supplementary problems

A. A *seminorm* on a vector space V is a map $V \to \mathbb{R}_{\ge 0}$ that satisfies all the conditions of a norm except that it can be zero for non-zero vectors.
   (a) Show that for any $f \in V'$, $\varphi(\underline{v}) = |f(\underline{v})|$ is a seminorm.
   (b) Construct a seminorm on $\mathbb{R}^2$ not of this form.
   (c) Let Φ be a family of seminorms on V which is pointwise bounded. Show that $\bar{\varphi}(\underline{v}) = \sup\{\varphi(\underline{v}) \mid \varphi \in \Phi\}$ is again a seminorm.

B. For $\underline{v} \in \mathbb{C}^n$ and $1 \le p \le \infty$ let $\|\underline{v}\|_p$ be as defined in class.
   (a) For $1 < p < \infty$ define $1 < q < \infty$ by $\frac{1}{p} + \frac{1}{q} = 1$ (also if $p = 1$ set $q = \infty$ and if $p = \infty$ set $q = 1$). Given $x \in \mathbb{C}$ let $y(x) = \frac{\bar{x}}{|x|}|x|^{p/q}$ (set $y = 0$ if $x = 0$), and given a vector $\underline{x} \in \mathbb{C}^n$ define a vector $\underline{y}$analogously.
      (i) Show that $\|\underline{y}\|_q = \|\underline{x}\|_p^{p/q}$.
      (ii) Show that $|\sum_{i=1}^n x_i y_i| = \|\underline{x}\|_p \|\underline{y}\|_q$
   (b) Now let $\underline{u}, \underline{v} \in \mathbb{C}^n$ and let $1 \le p \le \infty$. Show that $|\sum_{i=1}^n u_i v_i| \le \|\underline{u}\|_p \|\underline{v}\|_q$ (this is called *Hölder's inequality*).
   (c) Conlude that $\|\underline{u}\|_p = \max\left\{|\sum_{i=1}^n u_i v_i| \mid \|\underline{v}\|_q = 1\right\}$.
   (d) Show that $\|\underline{u}\|_p$ is a norm (hint: A(c)).
   (e) Show that $\lim_{p\to\infty}\|\underline{v}\|_p = \|\underline{v}\|_\infty$ (this is why the supremum norm is usually called the L^∞ norm).

C. Let $\{\underline{v}_n\}_{n=1}^\infty$ be a Cauchy sequence in a normed space. Show that $\{\|\underline{v}_n\|\}_{n=1}^\infty \subset \mathbb{R}_{\ge 0}$ is a Cauchy sequence.

D. Let X be a set. For $1 \le p < \infty$ set $\ell^p(X) = \{f\colon X \to \mathbb{C} \mid \sum_{x\in X}|f(x)|^p < \infty\}$, and also set $\ell^\infty(X) = \{f\colon X \to \mathbb{C} \mid f \text{ bounded}\}$.
   (a) Show that for $f \in \ell^p(X)$ and $g \in \ell^q(X)$ we have $fg \in \ell^1(X)$ and $|\sum_{x\in X} f(x)g(x)| \le \|f\|_p \|g\|_q$.
   (b) Show that $\ell^p(X)$ are subspaces of $\mathbb{C}^X$, and that $\|f\|_p = (\sum_{x\in X}|f(x)|^p)^{1/p}$ is a norm on $\ell^p(X)$
   (c) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence.

(d) Let $\{f_n\}_{n=1}^{\infty} \subset \ell^p(X)$ be a Cauchy sequence and let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f \in \ell^p(X)$.
(e) Let $\{f_n\}_{n=1}^{\infty} \subset \ell^p(X)$ be a Cauchy sequence. Show that it is convergent in $\ell^p(X)$.

E. Let V, W be normed vector spaces, equipped with the metric topology coming from the norm. Let $T \in \mathrm{Hom}_F(V,W)$. Show that the following are equivalent:
(1) T is continuous.
(2) T is continuous at zero.
(3) T is *bounded*: $\|T\|_{V\to W} < \infty$, that is: for some $C > 0$ and all $\underline{v} \in V$, $\|T\underline{v}\|_W \le C\|\underline{v}\|_V$.
Hint: the same idea is used in problem P1

F. Let V, W be normed spaces, and let $\mathrm{Hom}_{\mathrm{cts}}(V,W)$ be the set of bounded linear maps from V to W.
(a) Show that the operator norm is a norm on $\mathrm{Hom}_{\mathrm{cts}}(V,W)$.
(b) Suppose that W is complete with respects to its norm. Show that $\mathrm{Hom}_{\mathrm{cts}}(V,W)$ is also complete.

DEF The norm on $V^* \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathrm{cts}}(V,F)$ is called the *dual norm*.
(c) Let $V = \mathbb{R}^n$ and identify V^* with $\mathbb{R}^n$ via the basis of δ-functions. Show that the norm on V^* dual to the ℓ^1-norm is the ℓ^∞ norm and vice versa. Show that the ℓ^2-norm is self-dual.

G. (The completion) Let (X,d) be a metric space.
(a) Let $\{x_n\}, \{y_n\} \subset X$ be two Cauchy sequences. Show that $\{d(x_n,y_n)\}_{n=1}^{\infty} \subset \mathbb{R}$ is a Cauchy sequence.

DEF Let $(\tilde{X}, \tilde{d})$ denote the set of Cauchy sequences in X with the distance $\tilde{d}(\underline{x},\underline{y}) = \lim_{n\to\infty} d(x_n,y_n)$.
(b) Show that $\tilde{d}$ satisfies all the axioms of a metric except that it can be non-zero for distinct sequences.
(c) Show that the relation $\underline{x} \sim \underline{y} \iff \tilde{d}(\underline{x},\underline{y}) = 0$ is an equivalence relation.
(d) Let $\hat{X} = \tilde{X}/\sim$ be the set of equivalence classes. Show that $\tilde{d}\colon \tilde{X}\times\tilde{X} \to \mathbb{R}_{\ge 0}$ descends to a well-defined function $\hat{d}\colon \hat{X}\times\hat{X} \to \mathbb{R}_{\ge 0}$ which is a metric.
(e) Show that $(\hat{X}, \hat{d})$ is a complete metric space.

DEF For $x \in X$ let $\iota(x) \in \hat{X}$ be the equivalence class of the constant sequence x.
(f) Show that $\iota\colon X \to \hat{X}$ is an isometric embedding with dense image.
(g) (Universal property) Show that for any complete metric space (Y, d_Y) and any uniformly continuous $f\colon X \to Y$ there is a unique extension $\hat{f}\colon \hat{X} \to Y$ such that $\hat{f}\circ\iota = f$.
(h) Show that triples $(\hat{X}, \hat{d}, \iota)$ satisfying the property of (g) are unique up to a unique isomorphism.

Hint for D(d): Suppose that $\|f\|_p = \infty$. Then there is a finite set $S \subset X$ with $\left(\sum_{x\in S}|f(x)|^p\right)^{1/p} \ge \lim_{n\to\infty}\|f_n\| + 1$.