Math 412: Problem Set 4 (due 7/2/2014)

Practice

P1. Let U,V be vector spaces and let $A\subset U$, $B\subset V$ be subspaces.
  (a) "Naturally" embed $A\otimes B$ in $U\otimes V$.
  (b) Is $(U\otimes V)/(A\otimes B)$ isomorphic to $(U/A)\otimes(V/B)$?

P2. Let $(\cdot,\cdot)$ be a non-degenerate bilinear form on a finite-dimensional vector space U, definged by the isomorphism $g\colon U\to U'$ such that $(\underline{u},\underline{v})=(g\underline{u})(\underline{v})$.
  (a) For $T\in\mathrm{End}(U)$ define $T^\dagger=g^{-1}T'g$ where T' is the dual map. Show that $T^\dagger\in\mathrm{End}(U)$ satisfies $(\underline{u},T\underline{v})=\left(T^\dagger\underline{u},\underline{v}\right)$ for all $\underline{u},\underline{v}\in V$.
  (b) Show that $(TS)^\dagger=S^\dagger T^\dagger$.
  (c) Show that the matrix of $T^\dagger$ wrt any basis is the transpose of the matrix of T wrt that basis.

Bilinear forms

In problems 1,2 we assume 2 is invertible in F, and fix F-vector spaces V,W.

1. (Alternating pairings and symplectic forms) Let V,W be vector spaces, and let $[\cdot,\cdot]:V\times V\to W$ be a bilinear map.
  (a) Show that $(\forall\underline{u},\underline{v}\in V:[\underline{u},\underline{v}]=-[\underline{v},\underline{u}])\leftrightarrow(\forall\underline{u}\in V:[\underline{u},\underline{u}]=0)$ (Hint: consider $\underline{u}+\underline{v}$).
  DEF A form satisfying either property is *alternating*. We now suppose $[\cdot,\cdot]$ is alternating.
  (b) The *radical* of the form is the set $R=\{\underline{u}\in V\mid\forall\underline{v}\in V:[\underline{u},\underline{v}]=0\}$. Show that the radical is a subspace.
  (c) The form $[\cdot,\cdot]$ is called *non-degenerate* if its radical is $\{\underline{0}\}$. Show that setting $[\underline{u}+R,\underline{v}+R]\stackrel{\text{def}}{=}[\underline{u},\underline{v}]$ defines a non-degenerate alternating bilinear map $(V/R)\times(V/R)\to W$.
  RMK Note that you need to justify each claim, starting with "defines".

2. (Darboux's Theorem) Suppose now that V is finite-dimensional, and that $[\cdot,\cdot]:V\times V\to F$ is a non-degenerate alternating form.
  DEF The *orthogonal complement* of a subspace $U\subset V$ is a set $U^\perp=\{\underline{v}\in V\mid\forall\underline{u}\in U:[\underline{u},\underline{v}]=0\}$.
  (a) Show that $U^\perp$ is a subspace of V.
  (b) Show that the restriction of $[\cdot,\cdot]$ to U is non-degenerate iff $U\cap U^\perp=\{\underline{0}\}$.
  (*c) Suppose that the conditions of (b) hold. Show that $V=U\oplus U^\perp$, and that the restriction of $[\cdot,\cdot]$ to $U^\perp$ is non-degenerate.
  (d) Let $\underline{u}\in V$ be non-zero. Show that there is $\underline{u}'\in V$ such that $[\underline{u},\underline{u}]\neq 0$. Find a basis $\{\underline{u}_1,\underline{v}_1\}$to $U=\mathrm{Span}\{\underline{u},\underline{u}'\}$ in which the matrix of $[\cdot,\cdot]$ is $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$.
  (e) Show that $\dim_F V=2n$ for some n, and that V has a basis $\{\underline{u}_i,\underline{v}_i\}_{i=1}^n$ in which the matrix of $[\cdot,\cdot]$ is block-diagonal, with each 2×2 block of the form from (d).
  RECAP Only even-dimensional spaces have non-degenerate alternating forms, and up to choice of basis, there is only one such form.

Tensor product

4. (Preliminary step)
  (a) Construct a natural isomorphism $\mathrm{End}(U\otimes V)\to\mathrm{Hom}(U,U\otimes\mathrm{End}(V))$.
  (b) Generalize this to a natural isomorphism $\mathrm{Hom}(U\otimes V_1,U\otimes V_2)\to\mathrm{Hom}(U,U\otimes\mathrm{Hom}(V_1,V_2))$.

5. Let U,V be vector spaces with U finite-dimensional, and let $A \in \mathrm{Hom}(U, U\otimes V)$. Given a basis $\left\{\underline{u}_j\right\}_{j=1}^{\dim U}$ of U let $\underline{v}_{ij} \in V$ be defined by $A\underline{u}_j = \sum_i \underline{u}_i \otimes \underline{v}_{ij}$ and define $\mathrm{Tr} A = \sum_{i=1}^{\dim U} \underline{v}_{ii}$. Show that this definition is independent of the choice of basis.

6. (Inner products) Let U,V be inner product spaces (real scalars, say).
   (a) Show that $\langle u_1\otimes v_1, u_2\otimes v_2\rangle_{U\otimes V} \stackrel{\text{def}}{=} \langle u_1,u_2\rangle_U \langle v_1,v_2\rangle_V$ extends to an inner product on $U\otimes V$.
   (b) Let $A\in \mathrm{End}(U)$, $B\in\mathrm{End}(V)$. Show that $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$ (for a definition of the adjoint practice problem P2).
   (c) Let $P \in \mathrm{End}(U\otimes V)$, interpreted as an element of $\mathrm{Hom}(U, U\otimes \mathrm{End}(V))$ as in 1(b). Show that $(\mathrm{Tr}_U P)^\dagger = \mathrm{Tr}_U \left(P^\dagger\right)$.
   (*d) [Thanks to J. Karczmarek] Let $\underline{w}\in U\otimes V$ be non-zero, and let $P_{\underline{w}} \in \mathrm{End}(U\otimes V)$ be the orthogonal projection on $\underline{w}$. It follows from 3(c) that $\mathrm{Tr}_U P_{\underline{w}} \in \mathrm{End}(V)$ and $\mathrm{Tr}_V P_{\underline{w}} \in \mathrm{End}(U)$ are both Hermitian. Show that their non-zero eigenvalues are the same.

Supplementary problems

A. (Extension of scalars) Let $F\subset K$ be fields. Let V be an F-vectorspace.
   (a) Considering K as an F-vectorspace (see PS1), we have the tensor product $K\otimes_F V$ (the subscript means "tensor product as F-vectorspaces"). For each $x\in K$ defining a $x(\alpha\otimes\underline{v}) \stackrel{\text{def}}{=} (x\alpha)\otimes\underline{v}$. Show that this extends to an F-linear map $K\otimes_F V \to K\otimes_F V$ giving $K\otimes_F V$ the structure of a K-vector space. This construction is called "extension of scalars"
   (b) Let $B\subset V$ be a basis. Show that $\{1\otimes\underline{v}\}_{\underline{v}\in B}$ is a basis for $K\otimes_F V$ as a K-vectorspace. Conclude that $\dim_K(K\otimes_F V) = \dim_F V$.
   (c) Let $V_N = \mathrm{Span}_{\mathbb{R}}\left(\{1\}\cup\{\cos(nx),\sin(nx)\}_{n=1}^N\right)$. Then $\frac{d}{dx}\colon V_N\to V_N$ is not diagonable. Find a different basis for $\mathbb{C}\otimes_{\mathbb{R}} V_N$ in which $\frac{d}{dx}$ is diagonal. Note that the elements of your basis are not "pure tensors", that is not of the form $af(x)$ where $a\in\mathbb{C}$and $f=\cos(nx)$ or $f=\sin(nx)$.

B. DEF: An *F-algebra* is a triple $(A, 1_A, \times)$ such that A is an F-vector space, $(A, 0_A, 1_A+, \times)$ is a ring, and (compatibility of structures) for any $a\in F$ and $x,y\in A$ we have $a\cdot(x\times y) = (a\cdot x)\times y$. Because of the compatibility from now on we won't distinguish the multiplication in A and scalar multiplication by elements of F.
   (a) Verify that $\mathbb{C}$ is an $\mathbb{R}$-algebra, and that $M_n(F)$ is an F-algebra for all F.
   (b) More generally, verify that if R is a ring, and $F\subset R$ is a subfield then R has the structure of an F-algebra. Similarly, that $\mathrm{End}_F(V)$ is an F-algebra for any vector space V.
   (c) Let A,B be F-algebras. Give $A\otimes_F B$ the structure of an F-algebra.
   (d) Show that the map $F\to A$ given by $a\mapsto a\cdot 1_A$ gives an embedding of F-algebars $F\hookrightarrow A$.
   (e) (Extension of scalars for algebras) Let K be an extension of F. Give $K\otimes_F A$ the structure of a K-algebra.
   (f) Show that $K\otimes_F \mathrm{End}_F(V) \simeq \mathrm{End}_K(K\otimes_F V)$.

C. The *center* $Z(A)$ of a ring is the set of elements that commute with the whole ring.
   (a) Show that the center of an F-algebra is an F-subspace, containing the subspace $F\cdot 1_A$.
   (b) Show that the image of $Z(A)\otimes Z(B)$ in $A\otimes B$ is exactly the center of