MATH 253 – WORKSHEET 32
SPHERICAL COORDINATES

(1) Express the following surfaces in spherical coordinates.
 (a) The sphere of radius 2 about the origin.
 Solution: $\boxed{\rho \le 2}$
 (b) The "double cone" $z^2 = x^2 + y^2$.
 Solution: This reads $\rho^2 \cos^2 \phi = r^2 = \rho^2 \sin^2 \phi$, that is $|\tan \phi| = 1$, so $\boxed{\phi = \frac{\pi}{4}, \phi = \frac{3\pi}{4}}$.
 (c) The paraboloid $z = x^2 + y^2$.
 Solution: $\rho \cos \phi = \rho^2 \sin^2 \phi$ so $\rho = \frac{\cos \phi}{\sin^2 \phi}$.

(2) Let B be the ball of radius 1 about the origin. Evaluate $\iiint_B e^{-\left(x^2+y^2+z^2\right)^{3/2}} \, \mathrm{d}V$.
Solution: The domain is $\rho \le 1$, so the integral factors in spherical coordinates:

$$\begin{aligned}
\iiint_B e^{-\left(x^2+y^2+z^2\right)^{3/2}} \, \mathrm{d}V &= \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{\phi=0}^{\phi=\pi} \sin\phi \, \mathrm{d}\phi \int_{\rho=0}^{\rho=1} \rho^2 \, \mathrm{d}\rho e^{-\rho^3} \\
&= \left(\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta\right) \left(\int_{\phi=0}^{\phi=\pi} \sin\phi \, \mathrm{d}\phi\right) \left(\int_{\rho=0}^{\rho=1} \rho^2 \, \mathrm{d}\rho e^{-\rho^3}\right) \\
&= (2\pi) \left[-\cos\phi\right]_{\phi=0}^{\phi=\pi} \left[-\frac{1}{3} e^{-\rho^3}\right]_{\rho=0}^{\rho=1} \\
&= (2\pi)(2) \left(\frac{1-e^{-1}}{3}\right) = \frac{4\pi}{3}\left(1 - \frac{1}{e}\right).
\end{aligned}$$

(3) Describe the following regions in words, then set up integration in spherical coordinates:
 (a) $E = \left\{(x, y, z) \mid x, y, z \ge 0, \, x^2 + y^2 + z^2 \le 9\right\}$
 Solution: This is one eighth of the ball of radius 3. The ball is defined by $\rho \le 3$. That the points are in the positive quadrant is equivalent to $0 \le \theta \le \frac{\pi}{2}$ (think polar coordinates). That the points have $z \ge 0$ is equivalent to $0 \le \phi \le \frac{\pi}{2}$. In summary, the integral would read

$$\int_{\theta=0}^{\theta=\pi/2} \mathrm{d}\theta \int_{\phi=0}^{\phi=\pi/2} \sin\phi \, \mathrm{d}\phi \int_{\rho=0}^{\rho=3} \rho^2 \, \mathrm{d}\rho f$$

 (b) $E = \left\{(x, y, z) \mid x^2 + y^2 + (z-1)^2 \le 1\right\}$
 Solution: This is the ball of radius 1 about $(0, 0, 1)$. The condition is equivalent to $x^2 + y^2 + (z^2 - 2z + 1) \le 1$, that is $x^2 + y^2 + z^2 \le 2z$, which reads $\rho^2 \le 2\rho\cos\phi$, or $\rho \le 2\cos\phi$. Since the ball is above the xy plane, we have $0 \le \phi \le \frac{\pi}{2}$, so the integral is

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{\phi=0}^{\phi=\pi/2} \sin\phi \, \mathrm{d}\phi \int_{\rho=0}^{\rho=2\cos\phi} \rho^2 \, \mathrm{d}\rho f \, .$$

 If one wants to integrate $\mathrm{d}\rho$ first, then ρextends from 0 to 2 (the point most distant from the origin is the north point of the ball, at $(0, 0, 2)$). Then ϕ must satisfy $0 \le \phi \le \frac{\pi}{2}$ and $\cos\phi \ge \frac{\rho}{2}$, so the integral reads

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{\rho=0}^{\rho=2} \rho^2 \, \mathrm{d}\rho \int_{\phi=0}^{\phi=\cos^{-1}(\rho/2)} \sin\phi \, \mathrm{d}\phi f \, .$$

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Cylindrical or Spherical?

(1) Let E be the "dimple" inside the sphere $x^2+y^2+z^2=2$ and above the paraboloid $z=x^2+y^2$. Set up integration on it in spherical and cylindrical coordinates.

Cylindrical: We need points (x,y,z) below the upper hemisphere, that is belog the graph of $z=\sqrt{2-x^2-y^2}$, and above the graph of the paraboloid. In cylindrical coordinates we have $x^2+y^2=r^2$ so this reads: $r^2 \le z \le \sqrt{2-r^2}$. What about r,θ? The problem is clearly invariant under rotation around z-axis, so no constraint on θ. For r, at the origin we have $r=0$ and the largest circle in our "dimple" is at the intersection of the paraboloid and the sphere, that is on the circle of radius R where $R^2+(R^2)^2=2$ (plugging in $z=r^2$ into the equation $z^2+r^2=2$ of the sphere). The last equation can be rearranged to $\left(R^2\right)^2+R^2-2=0$ and factors as $\left(R^2+2\right)\left(R^2-1\right)=0$ which has the unique positive root $R=1$. It follows that $0\le r\le 1$, so we have

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=1} r\,\mathrm{d}r \int_{z=r^2}^{z=\sqrt{2-r^2}} \mathrm{d}z f\,.$$

Cylindrical, other order: In our dimple the z-range is $0\le z\le 2$ (from origin to north pole of sphere), and given z we have $r\le\sqrt{z}$ and $r\le\sqrt{2-z^2}$. Note that the two constraints point in the same direction, so we take the minimum. If $z\le 1$ then $\sqrt{z}\le\sqrt{2-z^2}$ (the plane of height z exits the dimple at the cone). If $z\ge 1$ then $\sqrt{2-z^2}\le\sqrt{z}$ (the plane at height z exits the dimple at the sphere). The integral is then

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{z=0}^{z=1} \mathrm{d}z \int_{r=0}^{r=\sqrt{z}} r\,\mathrm{d}r f + \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{z=1}^{z=2} \mathrm{d}z \int_{r=0}^{r=\sqrt{2-z^2}} r\,\mathrm{d}r f\,.$$

Spherical: We need to decide if a point (ρ,θ,ϕ) is inside the sphere and above the paraboloid. Nothing depends on θ, and to be inside the sphere simply means $\rho\le\sqrt{2}$. To be above the paraboloid means $z\ge r^2$ so $\rho\cos\phi\ge(\rho\sin\phi)^2$ or $\rho\le\frac{\cos\phi}{\sin^2\phi}$. So we have the same problem as in the second cylindrical case: for small ϕ (near the north pole) the radial line ends on the sphere. For larger ϕ (near the xy plane) the radial line ends on the paraboloid instead. The changeover occurs on the circle of intersection, which is at $z=1, r=1$ so at $\tan\phi=1$ and $\phi=\frac{\pi}{4}$. The integral is then

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{\phi=0}^{\phi=\pi/4} \sin\phi\,\mathrm{d}\phi \int_{\rho=0}^{\rho=\sqrt{2}} \rho^2\,\mathrm{d}\rho f + \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{\phi=\pi/4}^{\phi=\pi/2} \sin\phi\,\mathrm{d}\phi \int_{\rho=0}^{\rho=\frac{\cos\phi}{\sin^2\phi}} \rho^2\,\mathrm{d}\rho f\,.$$

For this we also used that $0\le\phi\le\frac{\pi}{2}$ since we are above the xy plane

Disucssion: Cylindrical was easiest since we didn't need to break the domain in two.

(2) Let E be the region above the cone $3z=\sqrt{x^2+y^2}$ and below the plane $z=\frac{1}{2}$. Set up integration on it.

Cylindrical: Symmetry uner rotation means there is no constraint on θ. Being between the cone and the plane reads $\frac{r}{3}\le z\le\frac{1}{2}$. The largest radius is at the base of the cone, when $z=\frac{1}{2}$ and hence $r=\frac{3}{2}$, so the integral reads

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=3/2} r\,\mathrm{d}r \int_{z=r/3}^{z=1/2} \mathrm{d}z f\,.$$

Cylindrical, other order: We can instead interpret the constraint as $r\le 3z$, so the integral is also

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{z=0}^{z=1/2} \mathrm{d}z \int_{r=0}^{r=3z} r\,\mathrm{d}r f\,.$$

Spherical: Points on the cone have $\tan\phi=\frac{r}{z}=3$, so being above the cone means $0\le\phi\le\tan^{-1}(3)$. The plane $z=\frac{1}{2}$ is $\rho\cos\phi=\frac{1}{2}$, so the integral is

$$\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{\phi=0}^{\phi=\tan^{-1}(3)} \sin\phi\,\mathrm{d}\phi \int_{\rho=0}^{\rho=\frac{1}{2\cos\phi}} \rho^2\,\mathrm{d}\rho f\,.$$

Disucssion: Now there is no obviuos advantage to either coordiante system; the choice will depend on f.