MATH 253 – WORKSHEET 24
MORE INTEGRATION IN POLAR COORDINATES

(1) Find the volume of the solid lying above the xy-plane, below the paraboloid $z = x^2 + y^2$ and inside the cylinder $(x-1)^2 + y^2 = 1$.
  (a) We found last time the set of points in the plane lying inside the cylinder is $D = \{(r, \theta) \mid r \le 2\cos\theta\}$. Find $f(r, \theta)$ describing the height of the solid above each such point.

  (b) Calculate the volume of the solid, that is $\iint_D f(r, \theta)\, \mathrm{d}A$.

(2) In this problem we will find the electrical field due to a sheet of charge. Suppose we have an infinite conducting plate in the xy plane, containing σ units of charge per unit area. The electrical field due to the plate must point vertically (why?), and can only depend on the height above the plate.
  (a) Consider a small part of the plate of area ΔA near the point $(x, y, 0)$. What is the charge Δq in this small part?

  (b) By the inverse square law, the electrical field at $(0, 0, z)$ due to the charge near $(x, y, 0)$ is given by the vector $\frac{k\Delta q}{|v|^3}\vec{v}$ where $\vec{v}$ is the vector between the two points. Express the vertical component of this vector as a function of (x, y).
  Solution: We have $\vec{v} = \langle -x, -y, z\rangle$ so $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$ and the projection of $\vec{v}$ on the vertical axis is z. In other words, we have $\Delta E_z \approx \frac{k\sigma z}{(x^2+y^2+z^2)^{3/2}}\Delta A$.

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(c) Express the electrical field at $(0, 0, z)$ by an integral.

(d) Evaluate the integral.

(e) Can you find a function $\phi(x, y, z)$ ("*Electric potential*") such that $-\vec{\nabla}\phi = \vec{E}$?

(3) In this problem we will find the area under the "bell curve". Let $I = \int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$, and let $J = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A$ (integral over the whole plane).

(a) Using an iterated integral in the xy coordinates relate J to I.

(b) Switch to polar coordinates and evaluate J.

(c) Given $\sigma > 0$ find a number Z such that $\int_{-\infty}^{+\infty} \left(\frac{1}{Z} e^{-x^2/2\sigma^2}\right) \mathrm{d}x = 1$.

(4) The electric potential at a point Z due to a charge q at the point X is $\frac{kq}{|\overrightarrow{XZ}|}$. Find the electrical potential at height z above the middle of a square plate of side length $2a$, if the charge density is σ.