MATH 253 – WORKSHEET 24
MORE INTEGRATION IN POLAR COORDINATES

(1) Find the volume of the solid lying above the xy-plane, below the paraboloid $z = x^2 + y^2$ and inside the cylinder $(x-1)^2 + y^2 = 1$.

(a) We found last time the set of points in the plane lying inside the cylinder is $D = \{(r,\theta) \mid r \le 2\cos\theta\}$. Find $f(r,\theta)$ describing the height of the solid above each such point.
Solution: $f(r,\theta) = x^2 + y^2 = r^2$.

(b) Calculate the volume of the solid, that is $\iint_D f(r,\theta)\,\mathrm{d}A$.
Solution 1: Natural to slice in radial lines, that is for each θ integrate over $0 \le r \le 2\cos\theta$. The bounding circle of D is tangent to the y-axis at the origin, so θ ranges from $-\frac{\pi}{2}$ (downward) to $\frac{\pi}{2}$ (upward). The integral is thus

$$\begin{aligned}
\int_{\theta=-\pi/2}^{\theta=\pi/2} \mathrm{d}\theta \int_{r=0}^{r=2\cos\theta} r^2 r\,\mathrm{d}r &= \int_{\theta=-\pi/2}^{\theta=\pi/2} \mathrm{d}\theta \left[\frac{r^4}{4}\right]_{r=0}^{r=2\cos\theta} \\
&= 4\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^4\theta\,\mathrm{d}\theta \\
&= 4\int_{\theta=-\pi/2}^{\theta=+\pi/2} \left(\frac{1}{2}(1+\cos(2\theta))\right)^2 \mathrm{d}\theta \\
&\overset{\alpha=2\theta}{=} \frac{1}{2}\int_{\alpha=-\pi}^{\alpha=\pi} \left(1 + 2\cos\alpha + \cos^2\alpha\right)\mathrm{d}\alpha \\
&= \frac{1}{2}\left(1 + 0 + \frac{1}{2}\right)2\pi = \frac{3}{2}\pi\,.
\end{aligned}$$

Here we used: (1) The half-angle formula $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$ and (2) That the average of $\cos\alpha$ over a full revolution is zero, while the average of $\cos^2\alpha$ is $\frac{1}{2}$ (since $\cos^2\alpha + \sin^2\alpha = 1$). An alternative to 2 is using the half-angle formula again: $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$ and now integrate from $-\pi$ to π.
Solution 2: Let's slice in concentric circles. The largest r value is 2 (when $\theta = 0$ this is possible), so we can also write the integral as

$$\begin{aligned}
\int_{r=0}^{r=2} r^3\,\mathrm{d}r \int_{\theta=-\arccos\frac{r}{2}}^{\theta=-\arccos\frac{r}{2}} \mathrm{d}\theta &= 2\int_0^2 r^3 \arccos\left(\frac{r}{2}\right)\mathrm{d}r \\
&\overset{r/2=\cos u}{=} 32\int_{u=\pi/2}^{u=0} \left(\cos^3 u\right)(u)(-\sin u)\,\mathrm{d}u \\
&\overset{\text{by parts}}{=} 8\left[-u\cdot\cos^4 u\right]_{u=0}^{u=\pi/2} + 8\int_{u=0}^{u=\pi/2} \cos^4 u\,\mathrm{d}u \\
&= 8\int_0^{\pi/2} \cos^4 u\,\mathrm{d}u
\end{aligned}$$

amd from now on we can continue as above (since cos is even this is equal to $4\int_{-\pi/2}^{+\pi/2}\cos^4 u\,\mathrm{d}u$).

(2) In this problem we will find the electrical field due to a sheet of charge. Suppose we have an infinite conducting plate in the xy plane, containing σ units of charge per unit area. The electrical field due to the plate must point vertically (why?), and can only depend on the height above the plate.

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(a) Consider a small part of the plate of area ΔA near the point $(x, y, 0)$. What is the charge Δq in this small part?

Solution 1: $\Delta q = \sigma \Delta A$ (charge/unit area $\times$ area).

(a) By the inverse square law, the electrical field at $(0, 0, z)$ due to the charge near $(x, y, 0)$ is given by the vector $\frac{k\Delta q}{|v|^3}\vec{v}$ where $\vec{v}$ is the vector between the two points. Express the vertical component of this vector as a function of (x, y).

Solution: We have $\vec{v} = \langle -x, -y, z\rangle$ so $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$ and the projection of $\vec{v}$ on the vertical axis is z. In other words, we have $\Delta E_z \approx \frac{k\sigma z}{(x^2+y^2+z^2)^{3/2}}\Delta A$.

(b) Express the electrical field at $(0, 0, z)$ by an integral.

Solution: Summing over the contributions from the whole plate, we get

$$E_z = \iint_{\mathbb{R}^2} \frac{k\sigma z}{(x^2 + y^2 + z^2)^{3/2}}\,\mathrm{d}A$$

(c) Evaluate the integral.

Solution: In polar coordinates

$$\begin{aligned} &= k\sigma z \int_{r=0}^{r=\infty} \frac{r\,\mathrm{d}r}{(r^2 + z^2)^{3/2}} \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \\ &= k\sigma z\,(2\pi) \left[\frac{1}{2}(-2)\left(r^2 + z^2\right)^{-1/2}\right]_{r=0}^{r\to\infty} \\ &= k\sigma z\,(2\pi)\,\frac{1}{z} = 2\pi k\sigma\,. \end{aligned}$$

Notes: (1) For the whole plane we go over all angles $(0 \le \theta \le 2\pi)$ and all radii $(0 \le r < \infty)$. (2) The derivative of $\left(r^2 + z^2\right)^{-1/2}$ is $-\frac{1}{2}\frac{2r}{(r^2+z^2)^{3/2}}$ which is our integrand up to constant (3) As $r \to \infty$ $\frac{1}{\sqrt{r^2+z^@}} \to 0$.

(d) Can you find a function $\phi(x, y, z)$ ("*Electric potential*") such that $-\vec{\nabla}\phi = \vec{E}$?

Solution: $\vec{E}(x, y, z) = 2\pi k\sigma\,\langle 0, 0, 1\rangle$ since the by rotational symmetry the field must point up or down. Since $\vec{\nabla}\phi$ is perpendicular to the level sets, we see that the level sets must be planes parallel to the xy plane, so ϕ can depend on z alone. Then $\vec{\nabla}\phi = \left\langle 0, 0, \frac{\partial\phi}{\partial z}\right\rangle$ and if this is constant we see that ϕ must be propotional to z, specifically that $\phi(x, y, z) = -2\pi k\sigma z$.

(3) In this problem we will find the area under the "bell curve". Let $I = \int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x$, and let $J = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A$ (integral over the whole plane).

(a) Using an iterated integral in the xy coordinates relate J to I.

Solution:

$$\begin{aligned} \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}A &= \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}y e^{-x^2-y^2} = \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2} \int_{-\infty}^{+\infty} \mathrm{d}y e^{-y^2} \\ &= \left(\int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2}\right)\left(\int_{-\infty}^{+\infty} \mathrm{d}y e^{-y^2}\right) = I^2\,. \end{aligned}$$

Note: realizing that $\int_{-\infty}^{+\infty} \mathrm{d}y e^{-y^2} = \int_{-\infty}^{+\infty} \mathrm{d}x e^{-x^2} = I$ wasn't easy. We are slowly working on developing the needed mental flexibility.

(b) Switch to polar coordinates and evaluate J.

Solution: $J = \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int_{r=0}^{r=\infty} r\,\mathrm{d}r e^{-r^2} = 2\pi\frac{1}{2}\int_{u=0}^{\infty} e^{-u}\,\mathrm{d}u = \pi$ where we substituted $u = r^2$ and used $\int_0^{\infty} e^{-u}\,\mathrm{d}u = 1$.

(c) Given $\sigma > 0$ find a number Z such that $\int_{-\infty}^{+\infty}\left(\frac{1}{Z}e^{-x^2/2\sigma^2}\right)\mathrm{d}x = 1$.

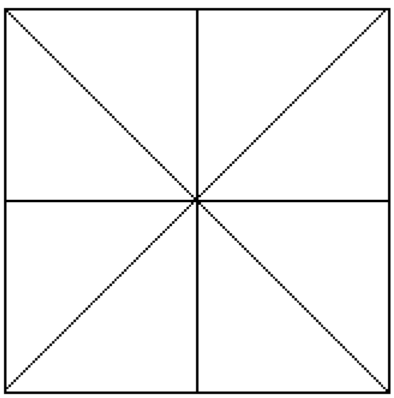
Solution: $\int_{-\infty}^{+\infty}\left(\frac{1}{Z}e^{-x^2/2\sigma^2}\right)\mathrm{d}x \overset{u=x/\sqrt{2}\sigma}{=} \frac{1}{Z}\sqrt{2}\sigma\int_{-\infty}^{+\infty} e^{-u^2}\,\mathrm{d}u = \frac{1}{Z}\sqrt{2}\sigma\sqrt{\pi} = \frac{\sqrt{2\pi\sigma^2}}{Z}$ so we need to choose $Z = \sqrt{2\pi\sigma^2}$.

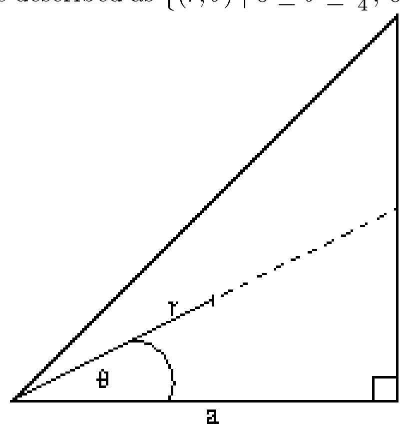
(4) The electric potential at a point Z due to a charge q at the point X is $\frac{kq}{|\overrightarrow{XZ}|}$. Find the electrical potential at height z above the middle of a square plate of side length $2a$, if the charge density is σ.

Solution: Parametrize the points on the plate as $\{(x, y, 0) \mid -a \leq x, y \leq a\}$. Then the distance to the point $(0, 0, z)$ above the middle of the plate is $\sqrt{x^2 + y^2 + z^2}$, so the integral is

$$\phi(z) = \iint_{[-a,a]^2} \frac{k\sigma \,\mathrm{d}A}{\sqrt{x^2 + y^2 + z^2}}$$

(recall that $\sigma \,\mathrm{d}A$ is the infinitesimal charge at the area element
dA). We convert this integral to polar coordinates. For this we divide the plate into eight triangles as in the left part of the figure below. Each of those has the same contribution (since any two differ only by changing the sign of x or y or both, or by exchanging x and y). So the total potential is eight times the potential created by the triangle magnified in the right-hand side. We parameterize the point (x, y) instead by (r, θ) as in the figure. Then $\left(x^2 + y^2 + z^2\right)^{-1/2} = \left(r^2 + z^2\right)^{-1/2}$. Also, we can see that r is at least zero, and at most the length of the hypotenuse in the right triangle below with angle θ and side a. It follows that the triangle can be described as $\left\{(r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{4},\, 0 \leq r \leq \frac{a}{\cos\theta}\right\}$.





The integral is then

$$\begin{aligned}
8k\sigma \int_{\theta=0}^{\theta=\frac{\pi}{4}} \mathrm{d}\theta \int_{r=0}^{r=\frac{a}{\cos\theta}} \frac{r\,\mathrm{d}r}{\left(r^2+z^2\right)^{-1/2}} &= 8k\sigma \int_{\theta=0}^{\theta=\frac{\pi}{4}} \mathrm{d}\theta \left[\left(r^2+z^2\right)^{1/2}\right]_{r=0}^{r=a/\cos\theta} \\
&= 8k\sigma \int_{\theta=0}^{\theta=\frac{\pi}{4}} \mathrm{d}\theta \left[\sqrt{\frac{a^2}{\cos^2\theta} + z^2} - z\right] \\
&= 8k\sigma \int_{\theta=0}^{\theta=\frac{\pi}{4}} \sqrt{\frac{a^2}{\cos^2\theta} + z^2}\,\mathrm{d}\theta - 2\pi k\sigma z\,.
\end{aligned}$$

Remark. The solution up to here was very difficult (many ideas were needed, and using polar coordinates on a triangle would not occur to anyone), but within the scope of 253. Actually calculating the remaining integral is not.

We now concentrate on the remaining integral. It seems natural to write the integrand as $\frac{\sqrt{a^2+z^2\cos^2\theta}}{\cos\theta} = \frac{\sqrt{a^2+z^2\cos^2\theta}}{\cos^2\theta}\cos\theta$ since $\cos\theta\,\mathrm{d}\theta = d(\sin\theta)$. Changing variables this way we see that

$$\int_{\theta=0}^{\theta=\frac{\pi}{4}} \sqrt{\frac{a^2}{\cos^2\theta} + z^2}\,\mathrm{d}\theta = \int_{u=0}^{u=1/\sqrt{2}} \frac{\sqrt{a^2+z^2-z^2u^2}}{1-u^2}\,\mathrm{d}u\,.$$

We can get rid of the square root by setting $u = \frac{\sqrt{a^2+z^2}}{z}\sin\alpha$ so that $\sqrt{(a^2+z^2)-z^2u^2} = \sqrt{a^2+z^2}\cos\alpha$, leading so the integral equals

$$\begin{aligned}
&= \frac{a^2+z^2}{z}\int_{u=0}^{u=1/\sqrt{2}} \frac{\cos\alpha}{1-\frac{a^2+z^2}{z^2}\sin^2\alpha}\cos\alpha\,\mathrm{d}\alpha \\
&= z(a^2+z^2)\int_{u=0}^{u=1/\sqrt{2}} \frac{\cos^2\alpha\,\mathrm{d}\alpha}{z^2-(a^2+z^2)\sin^2\alpha} \\
&= z\int_{u=0}^{u=1/\sqrt{2}} \frac{\cos^2\alpha}{\cos^2\alpha-\frac{a^2}{a^2+z^2}}\,\mathrm{d}\alpha\,.
\end{aligned}$$

For convenience set $A = \frac{a^2}{a^2+z^2}$. We then remove the $\cos^2\alpha$ from the numerator, in preparation for a trick

$$= z\int_{u=0}^{u=1/\sqrt{2}} \left[\frac{\cos^2\alpha - A}{\cos^2\alpha - A} + \frac{A}{\cos^2\alpha - A}\right]\mathrm{d}\alpha\,.$$

We can do the first integral, noting that $u=0$ corresponds to $\alpha=0$ and $u=1/\sqrt{2}$ corresponds to $\alpha = \arcsin\frac{z}{\sqrt{2(a^2+z^2)}}$ so

$$z\int_{u=0}^{u=1/\sqrt{2}} 1\,\mathrm{d}\alpha = z\arcsin\frac{z}{\sqrt{2(a^2+z^2)}}\,.$$

For the second integral, we divide through by $\frac{1}{\cos^2\alpha}$ and recall that $\frac{1}{\cos^2\alpha} = 1+\tan^2\alpha = \frac{\mathrm{d}(\tan\alpha)}{\mathrm{d}\alpha}$, so setting $t=\tan\alpha$ we have

$$\begin{aligned}
z\int_{u=0}^{u=1/\sqrt{2}} \frac{A\,\mathrm{d}\alpha}{\cos^2\alpha - A} &= z\int_{u=0}^{u=1/\sqrt{2}} \frac{1}{\frac{1}{A}-\frac{1}{\cos^2\alpha}}\frac{\mathrm{d}\alpha}{\cos^2\alpha} \\
&= zA\int_{u=0}^{u=1/\sqrt{2}} \frac{1}{\frac{1}{A}-(1+t^2)}\,\mathrm{d}t \\
&= z\int_{u=0}^{u=1/\sqrt{2}} \frac{1}{\frac{1-A}{A}-t^2}\,\mathrm{d}t\,.
\end{aligned}$$

Now,

$$\frac{1-A}{A} = \frac{1-\frac{a^2}{a^2+z^2}}{\frac{a^2}{a^2+z^2}} = \frac{z^2}{a^2}\,.$$

This integral therefore equals

$$\begin{aligned}
&= z\frac{a}{2z}\int_{u=0}^{u=1/\sqrt{2}} \left(\frac{1}{\frac{z}{a}+t} + \frac{1}{\frac{z}{a}-t}\right)\mathrm{d}t \\
&= \frac{a}{2}\left[\log\frac{\frac{z}{a}+t}{\frac{z}{a}-t}\right]_{u=0}^{u=1/\sqrt{2}}\,.
\end{aligned}$$

Now at $u=0$ we have $\alpha=0$ and thus $t=\tan 0 = 0$, at which point the logarithm vanishes. At $u=1/\sqrt{2}$ we have $\sin\alpha = \frac{z}{\sqrt{2(a^2+z^2)}}$ so

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{z^2}{2(a^2+z^2)}} = \sqrt{\frac{2a^2+z^2}{2a^2+2z^2}}\,.$$

Thus at $u=\frac{1}{\sqrt{2}}$ we have

$$t = \frac{z}{\sqrt{2a^2+z^2}}\,.$$

In conclusion,

$$\begin{aligned}
\int_{u=0}^{u=1/\sqrt{2}} \frac{\sqrt{a^2+z^2-z^2u^2}}{1-u^2}\,\mathrm{d}u &= z\arcsin\frac{z}{\sqrt{2(a^2+z^2)}} + \frac{a}{2}\log\frac{\frac{z}{a}+\frac{z}{\sqrt{2a^2+z^2}}}{\frac{z}{a}-\frac{z}{\sqrt{2a^2+z^2}}} \\
&= z\arcsin\frac{z}{\sqrt{2(a^2+z^2)}} + \frac{a}{2}\log\frac{\sqrt{2a^2+z^2}+a}{\sqrt{2a^2+z^2}-a}
\end{aligned}$$

and the potential is therefore

$$\phi(z) = 8k\sigma z\arcsin\frac{z}{\sqrt{2(a^2+z^2)}} + 4k\sigma a\log\frac{\sqrt{2a^2+z^2}+a}{\sqrt{2a^2+z^2}-a} - 2\pi k\sigma z\,.$$

The following is entirely unrelated to MATH 253.

Let's use Taylor expansions + physics intuition to check this kind of complicated answer. We'll examine the behaviour as $z\to 0$ and when $z\to\infty$ and check if it looks reasonable.

(1) As $z\to 0$, $z\arcsin\left(\frac{z}{\sqrt{2(a^2+z^2)}}\right)$ and $\log\frac{\sqrt{2a^2+z^2}+a}{\sqrt{2a^2+z^2}-a}$ are even functions of z, so jointly have a Taylor expansion $4k\sigma a\log\frac{\sqrt{2}+1}{\sqrt{2}-1}+Cz^2+\cdots$ for some C. It follows that for small z, our potential looks like

$$\phi(z)\approx 4k\sigma a\log\frac{\sqrt{2}+1}{\sqrt{2}-1} - 2\pi k\sigma z + \cdots$$

In this approximation, the closer we are to the plate the more it looks infinite, and indeed the non-constant term matches the answer to 2(d). What about the constant term? Recall that $\phi(z)$ in 2(d) was determined by its gradient, so we could put in it whatever constant we wanted – so in fact the answers match, except for higher-order corrections (terms in Taylor series).

(2) As $z\to\infty$,

$$\frac{z}{\sqrt{2(a^2+z^2)}} = \frac{1}{\sqrt{2}}\left(1+\frac{a^2}{z^2}\right)^{-1/2} \approx \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\frac{a^2}{z^2}\,.$$

Since the derivative of $\arcsin x$ at $x=\frac{1}{\sqrt{2}}$ is $\frac{1}{\sqrt{1-(1/\sqrt{2})^2}}=\sqrt{2}$, we make a linear approximation to see that as $z\to\infty$,

$$8k\sigma z\arcsin\frac{z}{\sqrt{2(a^2+z^2)}} \approx 8k\sigma z\left[\frac{\pi}{4}-\frac{1}{2}\frac{a^2}{z^2}\right] = 2\pi k\sigma z - \frac{4k\sigma a^2}{z}\,.$$

For the second term,

$$\begin{aligned}
\frac{\sqrt{2a^2+z^2}+a}{\sqrt{2a^2+z^2}-a} &= \frac{\left(1+\frac{2a^2}{z^2}\right)^{1/2}+\frac{a}{z}}{\left(1+\frac{2a^2}{z^2}\right)^{1/2}-\frac{a}{z}} \\
&\approx \frac{1+\frac{a}{z}+\frac{a^2}{z^2}}{1-\frac{a}{z}+\frac{a^2}{z^2}} \approx \frac{1+\frac{a}{z}}{1-\frac{a}{z}} = \frac{\left(1+\frac{a}{z}\right)^2}{1-\frac{a^2}{z^2}} \\
&\approx \left(1+\frac{a}{z}\right)^2
\end{aligned}$$

to first order in $\frac{1}{z}$ (that is, ignoring terms like $\frac{1}{z^2}$ or higher). Taking log we see that

$$4k\sigma a\log\frac{\sqrt{2a^2+z^2}+a}{\sqrt{2a^2+z^2}-a} \approx 8k\sigma a\log\left(1+\frac{a}{z}\right) \approx 8k\sigma a\frac{a}{z}\,.$$

Adding up everything we see that as $z\to\infty$

$$\begin{aligned}
\phi(z) &\approx 2\pi k\sigma z - \frac{4k\sigma a^2}{z} + \frac{8k\sigma a^2}{z} - 2\pi k\sigma z \\
&= \frac{4k\sigma a^2}{z} = k\frac{(\sigma(2a)^2)}{z}\,.
\end{aligned}$$

Now $(2a)^2$ is the area of the plate, so $\sigma(2a)^2$ is the total charge of the plate. In other words, as $z \to \infty$, $\phi(z)$ is approximately the potential due to a charge equal to the total charge of the plate, placed at the origin. This is what we expect (from very far away, the plate really looks like a point-like object).