Math 538: Problem Set 3 (due 15/4/2013)

Do a good amount of problems; choose problems based on what you already know and what you need to practice. Problems 2, 4, 6-8. 3 is useful for perspective.

Valuations and absolute values

1. Let R be an integral domain, $K(R)$ its field of fractions.
   (a) Show that any absolute value on $K(R)$ is uniquely determined by its restriction to R.
   (b) Let $|\cdot| : R \to \mathbb{R}_{\geq 0}$ be a map satisfying the definition of absolute values restricted to R. Show that it extends to an absolute value of $K(R)$.

2. Prove *Ostrowski's Theorem*: every absolute value on $\mathbb{Q}$ either discrete, equivalent to $|\cdot|_\infty$ or equivalent to $|\cdot|_p$ for some p.

3. Let F be a field, v a valuation on $F(t)$ which is trivial on F (for example, F might be a finite field).
   (a) Suppose that $v(f) \leq 0$ for all $f \in F[t]$. Show that $v(t) < 0$ and that up to rescaling $v = v_\infty$ where $v_\infty(f) = -\deg f$.
   (b) Suppose that $v(f) > 0$ for some $f \in F[t]$. Show there is an irreducible $p \in F[t]$ with that property, and that up to rescaling $v = v_p$ where $v_p(p^r \frac{f}{g}) = r$ where $(p, fg) = 1$.
   (c) (Product formula) Fix $q > 1$, and set for $f \in F(t)$, $p \in F(t)$ irreducible set $|f|_p = q^{-(\deg p)v_p(f)}$, $|f|_\infty = q^{\deg f}$. Show that for all $f \in \mathbb{Q}(t)^\times$,
   $$|f|_\infty \cdot \prod_{p \text{ monic irred}} |f|_p = 1\,.$$
   (d) Fix a transcendental $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{Q}_p$) and for $f \in \mathbb{Q}(t)$ set $|f| = |f(\alpha)|_\infty$ (or $|f| = |f(\alpha)|_p$). Show that this defines an absolute value on $\mathbb{Q}(t)$ which is not of the forms found in parts (a),(b).

3. Let (X,d) be an ultrametric space ($\forall x,y,z : d(x,z) \leq \max\{d(x,y), d(y,z)\}$)
   (a) ("all triangles are isoceles") Let $x,y,z \in X$ and suppose that $d(y,z) < d(x,y)$. Show that $d(x,z) = d(x,y)$.
   (b) Let B_1, B_2 be two closed balls in X. Show that one of the following holds: $B_1 \subset B_2$, $B_2 \subset B_1$, $B_1 \cap B_2 = \emptyset$.
   (c) Show that the completion of an ultrametric space is ultrametric.
   (d) Suppose that X is complete, and let $\{x_i\}_{i=1}^\infty \subset X$. Show that the sequence converges iff $\lim_{i\to\infty} d(x_i, x_{i+1}) = 0$.
   (e) (Calculus student's dream) Let $\{a_n\}_{n=0}^\infty \subset \mathbb{Q}_p$. Show that $\sum_{n=0}^\infty a_n$ converges iff $\lim_{n\to\infty} a_n = 0$.

On $\mathbb{Z}_p$

4. (Odds and ends)
   (a) Find $a_j \in \{0, \ldots, p-1\}$ such that $-1 = \sum_{j=0}^{\infty} a_j p^j$ in $\mathbb{Z}_p$.
   (b) Show that every $n \in \mathbb{Z}_{\geq 0}$ has a unique representation in the form $\sum_{j=0}^{J} a_j p^j$ (finite sum), $a_j \in \{0, \ldots, p-1\}$
   (c) Suppose p is odd. Show that every $n \in \mathbb{Z}$ has a unique representation in the form $\sum_{j=0}^{J} b_j p^j$ (finite sum), $b_j \in \left\{-\frac{p-1}{2}, \ldots, \frac{p-1}{2}\right\}$.
   (d) (Moving to $\mathbb{Q}_p$ is equivalent to localization) Show that $\mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)} = \left\{\frac{x}{s} \mid x, s \in \mathbb{Z},\, p \nmid s\right\}$.

5. Show that $\mathbb{Z}_p$, hence $\mathbb{Q}_p$ has the cardinality of the continuum.

Structure theory of $\mathbb{Q}_p$

Fix a field F and a non-archimedean absolute value $|\cdot|$ on F.

6. (Basic structure)
   (a) Let $R = \{x \in F \mid |x| \leq 1\}$. Show that R is a subring of F, and that it is a *valuation ring*: for all $x \in F^{\times}$ at least one of x, x^{-1} is in R. In particular, F is the field of fractions of R.
   (b) Show that $R^{\times} = \{x \in F \mid |x| = 1\}$.
   (c) (Analogue of "ring of integers") Show that R is *integrally closed:* if $\alpha \in F$ is a root of a monic $f \in R[x]$ then $\alpha \in R$ (hint: supposing $|\alpha| > 1$ calculate $|f(\alpha)|$ using 3(a)).
   (d) Show that $P = \{x \in R \mid |x| < 1\}$ is the unique maximal ideal of R and that it is open in R. In particular, R/P is a field and its natural topology is discrete.

7. (Discrete valuations) Suppose now that $\{|x| \mid x \in F^{\times}\}$ is a discrete subset of $\mathbb{R}_{>0}^{\times}$ (equivalently that $\{\log|x| \mid x \in F^{\times}\}$ is a discrete subgroup of $\mathbb{R}$).
   (a) Show that $\log|F^{\times}| = r\mathbb{Z}$ for some $r > 0$.
   (b) Show that $|\cdot|$ is non-archimedean.
   (c) ("Unique factorization") Let $\varpi \in P$ (read "pi", not "omega") have absolute value e^{-r} exactly. Show that the ideals of R are exactly $\left(\varpi^k\right)$ for $k \geq 0$ and (0).
   (d) ("Onion rings") Show that $\left\{x : |x| = e^{-kr}\right\} = \varpi^k R^{\times}$.
   (e) (Multiplicative group) For $k \geq 1$ show that $U_k = 1 + \varpi^k R$ is a compact and open subgroup of $R^{\times}$. Show that $U/U_1 \simeq (R/P)^{\times}$ and that for $k \geq 1$ $U_k/U_{k+1} \simeq (R/P, +)$.
   (f) ("Passing to the completion merely reorganizes the information") Compare (c)-(e) with the corresponding results for $\mathbb{Z}/p^K\mathbb{Z}$ we saw earlier.

8. (Representation of elements) In addition to the assumption of 7, suppose now that F is complete with respect to the absolute value.
   (a) ("Power series") Let $A \subset R$ be a set of representatives for R/P, with $0 \in A$. Show that every element of R has a unique representation in the form $\sum_{j=0}^{\infty} a_j \varpi^j$ where $a_j \in A$.
   (b) ("Laurent series") Show that every element of F has a unique representation in the form $\sum_{j=J}^{\infty} a_j \varpi^j$ where $a_j \in A$, $J \in \mathbb{Z}$ and $a_J \neq 0$.
   (c) Show that F is locally compact iff R is compact iff $R^{\times}$ is compact iff R/P is finite.