Math 223: Problem set 1 (due 12/9/12)

- Practice problems are selected from the text for solving at home.
- Numbered problems are the only ones for submissions. Problems marked * or ** may be unusually difficult. (§1.3 E3-4) refers problems 3,4 after section 1.3 of the textbook. RMK indicates a remark, not an exercise.
- Lettered problems as well as ones labelled SUPP are supplementary and not for submission; these generally cover additional ideas beyond the scope of the course.

Practice problems (recommended, but do not submit)

Section 1.2, problems 1-4, 8, 12-13, 17-19.
Section 1.3, problems 1-4, 8, 11, 16-17.

Linear equations

1. Find all solutions in real numbers to the following equations: (a) $5x+7=13$ (b) $\begin{cases} 5x+2y &= 3 \\ 6x+4y &= 2 \end{cases}$

   (c) $\begin{cases} 3x+2y &= a \\ 6x+4y &= a+1 \end{cases}$ (here a is a fixed real number; your answer will depend on a).

2. In each of the following problems (1) Convert the identity to a system of linear equations; (2) either exhibit a solution so that the identity holds true (in which case no proof is needed) or prove that no such solution exists.
   (a) $a\left(x^2+2x+1\right)+b\left(5x+3\right)+c\left(2\right)=7x^2-5x+3$;
   (b) $a\left(x^2-2x+1\right)+b\left(x-1\right)+c\left(x^2+5x\right)=x^2+2x+3$;
   (c) $a\left(x^2-2x+1\right)+b\left(x-1\right)+c\left(x^2-x\right)=x^2+2x+3$.

3. A matrix $A\in M_n(\mathbb{R})$ is called *skew-symmetric* if $A^t=-A$. Show that $A-A^t$ is skew-symmetric for all $A\in M_n(\mathbb{R})$. You may use the results of problems (§1.3 E3,4) if you wish.

Subspaces

4. In each case decide if the set is a subspace of the given space.
   (a) $V_1=\left\{f\in\mathbb{R}^{\mathbb{R}} \mid \forall t\neq 0: f(t)=2f(2t)\right\}, V_2=\left\{f\in\mathbb{R}^{\mathbb{R}} \mid \forall t\neq 0: f(t)=f(2t)+1\right\}$ in $\mathbb{R}^{\mathbb{R}}$.
   (b) Let $U_1=\left\{\underline{x}\in\mathbb{R}^3 \mid x_1+x_3-1=0\right\}, U_2=\left\{\underline{x}\in\mathbb{R}^3 \mid x_1-2x_2+x_3=0\right\}$,
   $U_3=\left\{\underline{x}\in\mathbb{R}^3 \mid x_1^2+x_2^2-x_3^2=0\right\}$ in $\mathbb{R}^3$.

5. Fix a vector space V.
   (a) Let $W\subset V$ be a subset. Show that W is a subspace of V if and only if the following two conditions hold:
      (1) $\underline{0}\in W$
      (2) For all $\underline{u},\underline{v}\in W$ and $a,b\in\mathbb{R}$we have $a\underline{u}+b\underline{v}\in W$.
   (b) Now let $W\subset V$ be a subspace. For any $n\geq 0$ let $\{\underline{w}_i\}_{i=1}^n\subset W$ be some vectors and let $\{a_i\}_{i=1}^n\subset\mathbb{R}$ be some scalars. Give an informal argument showing $\sum_{i=1}^n a_i\underline{w}_i=a_1\underline{w}_1+\cdots+a_n\underline{w}_n\in W$.
   (*c) Give a formal proof by induction on n.

More on 5(a) on the next page.

You may use the facts (to be proved in class) that for all $\underline{v} \in V$, $0 \cdot \underline{v} = \underline{0}$ and $(-1) \cdot \underline{v} = -\underline{v}$.

To get you started in 5(a): you need to verify all the properties in the definition of a vector space; one of them is that if $\underline{u}, \underline{v} \in W$ then $\underline{u} + \underline{v} = \underline{v} + \underline{u}$. Why is that true? It's true because $\underline{u}, \underline{v}$ are elements of V (W is contained in V after all), and V is assumed to be a vector space.

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6. (A chain of subspaces)
   (a) Show that the space of bounded functions on a set X,
   $$\ell^\infty(X) = \left\{ f \in \mathbb{R}^X \mid \text{There is } M \in \mathbb{R} \text{ so that for all } x \in X, \text{ we have } |f(x)| \leq M \right\},$$
   is a subspace of $\mathbb{R}^X$.
   (b) State (or reconstruct) Theorems from calculus to the effect that "the space of convergent sequences, $c = \{\underline{a} \in \mathbb{R}^\infty \mid \lim_{n\to\infty} a_n \text{ exists}\}$, is a subspace of $\ell^\infty(\mathbb{N})$".
   RMK If you haven't seen those theorems before you can write them down first and then confirm their existence in your calculus textbook or online. Don't forget that subspaces are subsets!
   (c) Show that the space of sequences of finite support, $\mathbb{R}^{\oplus\mathbb{N}} = \left\{ \underline{a} \in \mathbb{R}^{\mathbb{N}} \mid a_i \neq 0 \text{ for finitely many } i \right\}$, is a subspace of c. [now you need to know a little about convergent sequences]

**7. (§1.3 E19) Let V be a vector space and let W_1, W_2 be subspaces of V. Suppose that union $W_1 \cup W_2 = \{v \in V \mid v \in W_1 \text{ or } v \in W_2\}$ is a subspace of V (note that "or" includes the possibility that both assertions hold). Show that $W_1 \subset W_2$ or $W_2 \subset W_1$.

New spaces from old ones

8. Let V, W be two vector spaces. On the set of pairs $V \times W = \{(\underline{v}, \underline{w}) \mid \underline{v} \in V, \underline{w} \in W\}$ define $(\underline{v}_1, \underline{w}_1) + (\underline{v}_2, \underline{w}_2) = (\underline{v}_1 +_V \underline{v}_2, \underline{w}_1 +_W \underline{w}_2)$ and $a \cdot (\underline{v}_1, \underline{w}_1) = (a \cdot_V \underline{v}_1, a \cdot_W \underline{w}_1)$. Show that this endows $V \times W$ with the structure of a vector space. We will call this space the *external direct sum* of V, W and denote it $V \oplus W$.

9. Let W_1, W_2 be two subspaces of a vector space V.
   (a) Define their *internal sum* to be the set $W_1 + W_2 \stackrel{\text{def}}{=} \{\underline{w}_1 + \underline{w}_2 \mid \underline{w}_i \in W_i\}$. Show that $W_1 + W_2$ is a subspace of V.
   (b) Show that $W_1 \cap W_2 = \{\underline{0}\}$ if and only if every vector in $W_1 + W_2$ has a *unique* representation in the form $\underline{w}_1 + \underline{w}_2$. (Hint on next page)

RMK In the case the equivalent conditions of (b) hold, we say that $W_1 + W_2$ is the *internal direct sum* of W_1, W_2 and confusingly also denote this space $W_1 \oplus W_2$. We will show later that in this case the two "direct sums" produced by problems 8 and 9(b) are in some sense the same. In general it will be possible to tell from context which direct sum is intended.

Hint for 9 (easy direction): Given a non-zero $\underline{w} \in W_1 \cap W_2$ consider that $\underline{w} + (-\underline{w}) = \underline{0}$.

Hint for 9 (difficult direction): Suppose that $\underline{w}_1 + \underline{w}_2 = \underline{w}'_1 + \underline{w}'_2$. Rearrange this to get an equality of vectors, one from W_1 and the other from W_2.

Supplementary problems: abstractions

A. Write B^A for the set of all functions from the set A to the set B.
   (a) Let a' not be an element of A, and let $A' = A \cup \{a'\}$ be the set you get by adding a' to A. Construct a bijection between $B^{A'}$ and the set of pairs $B^A \times B = \left\{(f,b) \mid f \in B^A, b \in B\right\}$.
   (b) Suppose that A, B are finite sets. Show that $\#\left(B^A\right) = (\#B)^{(\#A)}$ where $\#X$ denotes the number of elements of a set X and on the right we have exponentiation of natural numbers. *Hint*: Induction on $\#A$.

   RMK Make sure to account for the corner cases where at least one the sets A, B is empty!

B. (Direct products and sums in general)
   (a) Let $\{V_i\}_{i\in I}$ be a family of vector spaces, and let $\prod_{i\in I} V_i$ (their *direct product*) denote the set $\{f : I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i\}$ (that is, the set of functions f with domain I such that $f(i)$ is an element of V_i for all i). For $f, g \in \prod_{i\in I} V_i$ and $a, b \in \mathbb{R}$ define $af + bg$ by $(af + bg)(i) = af(i) + bg(i)$ (addition and multiplication in V_i). Show that this endows $\prod_{i\in I} V_{i\in I}$ with the structure of a vector space.
   (b) Continuing with the same family, define the *support* of $f \in \prod_i V_i$ analogously to 7(d) and show that the *direct sum* $\bigoplus_{i\in I} V_i \stackrel{\text{def}}{=} \{f \in \prod_i V_i \mid \operatorname{supp}(f) \text{ is finite}\}$ is a subspace.
   (c) When all the V_i are equal to a fixed space V we sometimes write V^I for the direct product $\prod_{i\in I} V$, $V^{\oplus I}$ for the direct sum $\oplus_{i\in I} V$. Verify that this agrees with the notation in 7(c). What is V there?

Supplementary problems: fields

Notation: $\forall$ means "For all" and $\exists$ mean "there exists".

DEFINITION. A *field* is a triple $(F, +, \cdot)$ of a set F and two binary opetarations on F so that there are elements $0, 1 \in F$ for which:

$$\forall x, y, z \in F \;:\; x + y = y + x,\; (x+y)+z = x+(y+z),\; x + 0 = x,\; \exists x' : x + x' = 0$$
$$\forall x, y, z \in F \;:\; x \cdot y = y \cdot x,\; (x \cdot y) \cdot z = x \cdot (y \cdot z),\; x \cdot 1 = x,\; (x \neq 0) \Rightarrow \exists \tilde{x} : x \cdot \tilde{x} = 1$$
$$\forall x, y, z \in F : x \cdot (y + z) = x \cdot y + x \cdot z$$

C. (Elementary calculations) Let F be a field.
   (a) Let $0_1, 0_2$ be two elements of F which can be used in the definition above. By considering the sum $0_1 + 0_2$ show that $0_1 = 0_2$.
   (b) Let $x \in F$ and let $x'_1, x'_2 \in F$ be such that $x + x'_1 = x + x'_2 = 0$. Adding x'_1 to both sides conclude that $x'_1 = x'_2$. This element is usually denoted $-x$.
   (c) Let $x \in F$. Show that $0 \cdot x = 0$.
   (d) Similarly show that 1 and $\tilde{x}$ (usually denoted x^{-1}) are unique.
   (e) Show that if $xy = 0$ then $x = 0$ or $y = 0$.

D. Consider the set $\{0, 1\}$ with $0 \neq 1$. Define $1 + 1 = 0$, and define all other sums and products in this set as required by the definition above or by A(c). Show that the result is a field. Show that defining $1 + 1 = 1$ would not result in a field, and conclude that there is a unique field with two elemnets, denoted $\mathbb{F}_2$ from now on.

DEFINITION. A vector space over the field F has the same definition as given in class, except that the field of scalars $\mathbb{R}$ is replaced with F.

E. Let X be a set. To a subset $A \subset X$ associate its *indicator function* $1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$. Show that the map $A \mapsto 1_A$ gives a bijection between the *powerset* $\mathcal{P}(X) = \{A \mid A \subset X\}$ and the vector space $\mathbb{F}_2^X$. Show that under this identification addition in $\mathbb{F}_2^X$ maps to the operation *of symmetric difference* of sets, defined by $A\Delta B = \{x \mid x \in A \cup B, x \notin A \cap B\}$ (that is, $A\Delta B$ is the set of elements of X that are in *exactly one* of A, B but not both).

F. Let F be a field with finitely many elements. For an integer $n \geq 0$ write $\bar{n} = \sum_{i=1}^{n} 1_F$.
  (a) Show that $\bar{n} = \bar{m}$ for some $n > m > 0$ and conclude that $\bar{p} = 0_F$ for some positive integer p.
  (b) Show that the smallest positive p such that $\bar{p} = 0_F$ is a prime number. This is called the *characteristic* of F and denoted $\operatorname{char}(F)$.
  (*c) Show that $\{\bar{i} \mid 0 \leq i < \operatorname{char}(F)\}$ is a subfield of F, usually denoted the *prime field* of F.
  RMK We will later show that if F has characteristic p then its number of elements is of the form $q = p^f$ for some integer f.