MATHEMATICAL NOTATIONS

1. Logical remarks

"or" means *inclusive* or: the statement "$0 = 0$ or $1 = 1$" is a true statement.

$\forall x$ means "for all x", $\exists x$ means "there exists x"

2. Set theory

$\in$ denotes set membership (negation $\notin$): $1 \in \{1, 2, 3\}$ but $0 \notin \{1, 2, 3\}$. The empty set is denoted $\emptyset$.

For a property $\phi(x)$ of elements of a set A we write $\{x \in A \mid \phi(x)\}$ for the subset of A consisting of those elements satisfying ϕ (example: $\{x \in \mathbb{R} \mid x > 0\}$ is the set of positive reals).

Write $B \subset A$ or $B \subseteq A$ for *set containment*, the assertion that $x \in B \Rightarrow x \in A$ (example; $\{0\} \subset \{0, 1\} \subset \{0, 1, 2\}$ but $\{2, 3\} \not\subset \{0, 1, 2\}$.

Write $\mathcal{P}(A)$ for the *powerset* $\{B \mid B \subset A\}$. Example:

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

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2.1. Operations on sets.

Fix sets A, B. Write:

- $A \cup B$ for the *union* $\{x \mid x \in A \text{ or } x \in B\}$ [the elements appearing in at least one of the sets] (example: $\{0,1\} \cup \{1,2\} = \{0,1,2\}$)
- $A \cap B$ for the *intersection* $\{x \in A \mid x \in B\} = \{x \in B \mid x \in A\} = \{x \mid x \in A \text{ and } x \in B\}$ [the elements appearing in both of the sets] (example: $\{0,1\} \cap \{1,2\} = \{1\}$
- $A \times B$ for the *Cartesian product* $\{(a, b) \mid a \in A,\, b \in B\}$ [the set of pairs of elements].
- $A \setminus B$ for the *difference* $\{x \in A \mid x \notin B\}$ [the elements appearing in A but not B], $A \Delta B$ for the *symmetric difference* $(A \setminus B) \cup (B \setminus A)$ [the elements appearing in exactly one of the sets] (example: $\{0,1\} \setminus \{1,2\} = \{0\}$, $\{0,1\} \Delta \{1,2\} = \{0,2\}$).

2.2. Functions.

Given sets A, B we write $f : A \to B$ to mean that f is a function with domain A (i.e. defined for all $a \in A$) and so that $f(a) \in B$ for all $a \in A$. We sometime instead write this function by listing its values like so, writing f_a instead of $f(a)$:

$$\{f_a\}_{a \in A} .$$

Don't confuse this notation with the set notation.

For a set A write id_A for the *identity function on A*. That's the function $\mathrm{id}_A : A \to A$ such that $\mathrm{id}_A(a) = a$ for all a.

Note that functions are equal if the have the same domain, and if their values agree at every element of the domain.

- Given functions $f : A \to B$ and $g : B \to C$ we define their *composition* $g \circ f : A \to C$ to be the function $(g \circ f)(a) = g(f(a))$.
- Call a function $f : A \to B$ *injective* (or $1-1$) if $a \neq a'$ implies $f(a) \neq f(a')$.

- Call a function (or onto) if for every $b \in B$ there is $a \in A$ so that $f(a) = b$.
- Call a fucntion $f\colon A \to B$ *bijective* if it is injective and surjective.

Exercise. f is bijective iff there is $g\colon B \to A$ so that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

2.3. **"Family" set operations [will generally not appear in the course].** Fix a family $\{A_i\}_{i\in I}$ of sets (see function notation above) we write

- $\bigcup_{i\in I} A_i$ for the *union* $\{x \mid \exists i \in I : x \in A_i\}$
- $\bigcap_{i\in I} A_i$ for the *intersection*$\{x \mid \forall i \in I : x \in A_i\}$
- $\prod_{i\in I} A_i$ for the *Cartesian product* $\left\{f\colon I \to \bigcup_{i\in I} A_i \mid \forall i \in I : f(i) \in A_i\right\}$.

3. Linear algebra

3.1. **Summation notation.** Let V be a vector space, $\{v_i\}_{i=1}^N$ a sequence of elements of V. We set inductively: $\sum_{i=1}^0 \underline{v}_i = \underline{0}$ (the empty sum is by definition zero) after that $\sum_{i=1}^{n+1} \underline{v}_i \stackrel{\text{def}}{=} (\sum_{i=1}^n \underline{v}_i) + \underline{v}_{n+1}$. In other words:

$$\sum_{i=1}^{0} \underline{v}_i = \underline{0}, \quad \sum_{i=1}^{1} \underline{v}_i = \underline{v}_1, \quad \sum_{i=1}^{2} \underline{v}_i = \underline{v}_1 + \underline{v}_2, \quad \sum_{i=1}^{3} \underline{v}_i = (\underline{v}_1 + \underline{v}_2) + \underline{v}_3, \ \ldots$$

- Note that the value of the empty sum depends on the vector space under consideration!