MORE EXAMPLES

1. Vector spaces and subspaces

Examples of calculations in $\mathbb{R}^3$. The sum of the vectors $\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \\ -3 \end{pmatrix}$ is $\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix} + \begin{pmatrix} 5 \\ 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 5+5 \\ 7+9 \\ 8+(-3) \end{pmatrix} = \begin{pmatrix} 10 \\ 16 \\ 5 \end{pmatrix}$. Similarly, $-2 \cdot \begin{pmatrix} 5 \\ 9 \\ -3 \end{pmatrix} = \begin{pmatrix} -2(5) \\ -2(9) \\ -2(-3) \end{pmatrix} = \begin{pmatrix} -10 \\ -18 \\ 6 \end{pmatrix}$

Subspace / a non-subspace? (preparation for PS1 problem 4). In $\mathbb{R}^2$ consider the set $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 + x_2 = 1 \right\}$. This is not a subspace – for example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ belong to this set but their sum doesn't (the zero vector also isn't, and is the first thing you should check for). Also consider the set $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1^2 - x_2^2 = 0 \right\}$. This is not a subspace. For example, it contains $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ but not their sum $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. [Aside: note that if $x_1^2 - x_2^2 = 0$ then $(ax_1)^2 - (ax_2)^2 = 0$ also, so this set is closed under rescaling!).

Generally, you should expect a subspace to be defined by *linear* conditions not by polynomials of higher degree.

Something which is not a vector space. Consider the triple $(\mathbb{R}^2, +, \cdot)$ where $+$ is the usual addition of column vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} x_1 + y_1 \\ x_2 + y_1 \end{pmatrix}$, but $a \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} ax_1 \\ 0 \end{pmatrix}$ for all a, x_1, x_2. All properties of addition will be satisfied, the distributive laws still hold (check!) and $(ab) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a \left(b \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)$ will still be true. But according to the definition above we have $1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and this is not the same as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so the axiom about 1 does not hold.

Tips.

(1) To show something is a vector space from the definitions, you need to go through all the axioms and check all of them.
(2) Almost always, though, the candidate vector space is a candidate *subspace* of a known space, and it's enough to check closure under the operations (and non-emptyness!)

(3) To show something is *not* a vector space, it's enough to show one axiom that fails; no more work is required.

2. Linear dependence and independence

Linear independence of two vectors. We check that $\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \\ -3 \end{pmatrix}$ are independent in $\mathbb{R}^3$. We need to see if there's a non-zero solution to $a \begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix} + b \begin{pmatrix} 5 \\ 9 \\ -3 \end{pmatrix} = \underline{0}$. By definition of $\mathbb{R}^3$ this is equivalent to $\begin{pmatrix} 5a+5b \\ 7a+9b \\ 8a-3b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (we have converted out claim to a system of linear equations). The first equation now forces $b = -a$ while the second forces $b = -\frac{7}{9}a$ and the two are impossible at the same time unless $b = 0$ and $a = 0$, so the vectors are independent.
Linear dependence of a vector on three others. See problem 2 of PS3.
Linear dependence of a function on two others. Note that $\cos^2 x$ depends on 1 and $\cos 2x$ on any interval since $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.
Linear independence of an infinite set. In the $\mathbb{R}^\infty$ (the space of seqeuences) let $\underline{v}_k$ be the sequence consisting of 1 appearing k times and then all zeroes. So $\underline{v}_1 = (1,0,0,0,0,0,\cdots)$, $\underline{v}_2 = (1,1,0,0,0,0,\cdots)$, $\underline{v}_3 = (1,1,1,0,0,0,\cdots)$ and so on. We claim that $\{\underline{v}_k\}_{k=1}^\infty$ are independent.

- Let's see that $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are independent. Suppose that $a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 = \underline{0}$. Let's write the first 5 terms of each sequence. We get the equality:
$$(a_1 + a_2 + a_3, a_2 + a_3, a_3, 0, 0, \cdots) = (0,0,0,0,0,\cdots) .$$
Now looking at the third element we see $a_3 = 0$, and we can go back to the previous two to succesively show that $a_2 = 0$ and $a_1 = 0$.
- Compare this argument with that of PS1 problem 2(a).
- Now let's treat the general case. Suppose that some finite subset $S \subset \{\underline{v}_k\}_{k=1}^\infty$ is linearly dependent.. Let n be the largest such that $\underline{v}_n \in S$. Then $S \subset \{\underline{v}_k\}_{k=1}^n$ (n is largest in S). Linear dependence means there are coefficients a_k, not all zero, so that $\sum_{k=1}^n a_k\underline{v}_k = \underline{0}$. Now let $K \leq n$ be alrgest so that $a_K \neq 0$ (this exists since not all a_k are zero). Then $\sum_{k=1}^K a_k\underline{v}_k = \underline{0}$. Finally, consider the Kth element of the sequence in the identity. for $k < K$ the Kth element of $\underline{v}_k$ is zero (only k ones). It follows that the Kth element of $\sum_{k=1}^K a_k\underline{v}_k$ is a_K (since $\underline{v}_K$ has 1 at the Kth positiion). But this shows that $a_K = 0$, a contradiction.

Summary. To prove a finite set is linearly independent:

(1) Set up the system of linear equations $\sum_i a_i\underline{v}_i = \underline{0}$
(2) Try to show that all the a_i are zero.

To prove that an *infinite* set is linearly independent

(1) Consider an *arbitrary* finite subset S of the set.
(2) Now you're dealing with a finite problem.