223: QUESTIONS AND ANSWERS

1. Metamathmatics

Question 1. *How do I prove that something is unique?*

Suppose you need to prove "there is at most one kind of object of type T". Then start with "Suppose a and b are both objects of type T" and then try to show that $a = b$. This shows that there cannot be different objects of this type. If you want "there is exactly one object of type T" then, in addition to that, you also need to prove that some object of this type exists.

Question 2 (Difficulty in PS1)**.** *What is "if and only if", usually shortened iff?*

If P and Q are two assertion, we say that "P if and only if Q" if they are equivalent, in the sense that they both imply each other. If you are asked to prove this you need to prove both directions: both that P implies Q and that Q implies P.

For example, consider the following assertions about a vector $\underline{v} \in V$: f$P(\underline{v}) =$ "$\underline{v}$ is the zero vector" and $Q(\underline{v})$ is "$\underline{v} + \underline{v} = \underline{v}$". In class we proved that $P \Rightarrow Q$ (by plugging in zero in the equation) and that $Q \Rightarrow P$ (by subtracting $\underline{v}$ from both sides of the equation).

2. Notation

Question 3. *Is $\ell^\infty(X)$ the same as X?*

No. For example, if X is the interval $[0, 1]$ then $\ell^\infty(X)$ is the space of bounded functions on the interval $[0, 1]$. Try proving: if $X = \{1, 2, 3\}$ then $\ell^\infty(X) = \mathbb{R}^3$ since every function is bounded.

Question 4 (Difficulty in PS2)**.** *What does it mean for different expressions "to be equal/different as functions on a set"?*

The point is that an expression like $x^2 + 5$ or $\cos(3x)$ can be just a formal expression, but it is commonly used to describe a function (the function you get by evaluating the expression at various values of x). However, when you use it to describe a function the notation is missing *the domain* where the function is supposed to be defined. Consider the two polynomials $x(x-1)$ and $x(x-1)(x-2)$. They are certainly different polynomials, but they define the same function on the set $\{0, 1\}$. They define different functions on the set $\{0, 1, 2\}$. We say "$x(x-1) = x(x-1)(x-2)$ as functions on $\{0, 1\}$ but $x(x-1) \neq x(x-1)(x-2)$ as functions on $\{0, 1, 2\}$.

We prove in this course: let $A \subset \mathbb{R}$ be finite, and let $f : A \to \mathbb{R}$ be any function. Then there is a polynomial $p \in \mathbb{R}[x]$ of degree at most $\#A - 1$ so that $p = f$ as functions on A. Try proving the cases $A = \{a_1\}$, $A = \{a_1, a_2\}$ and $A = \{a_1, a_2, a_3\}$by hand.

3. Linear algebra

Question 5 (Difficulty in PS1). *What is the difference between a subset and a subspace?*

Consider the vector space $\mathbb{R}^2$. A subset of $\mathbb{R}^2$ means any collection of vectors belonging to $\mathbb{R}^2$, for example $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ \pi \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} x \\ x^2 \end{pmatrix} \mid x \in \mathbb{R} \right\}$ is a subset of $\mathbb{R}^2$. A *subspace* is a subset ssatisfying extra conditions (it must be non-empty, and closed under the vector space operations). For example, $\left\{ \begin{pmatrix} x \\ 2x \end{pmatrix} \mid x \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$.

Question 6. *Suppose $\underline{u}$, $\underline{v}$ are linearly independent. Aren't $0\underline{u}$ and $0\underline{u} + 0\underline{v}$ two different ways of writing the zero vector as a combination?*

No – we don't distinguish representations involving multiplication by zero. So both of the example representations are the same as the empty sum. Similarly, $2\underline{u}$ and $2\underline{u} + 0\underline{v}$ are considered the same representation of the vector $2\underline{u}$.

4. Complex numbers

Question 7. *Let $z \in \mathbb{C}$. Why do we write $\bar{z}z = |z|^2$ and not $\bar{z}z = |z^2|$?*

The first statement is the *definition* of the absolute value: we set $|z| = \sqrt{\bar{z}z}$. The second claim is true, but would be an annoying way to define the absolute value (to compute $|w|$ you'd need to find z for which $z^2 = w$).

5. Inner product spaces

Question 8. *Let V be an inner product space, and let $\underline{u}, \underline{u}' \in V$. Suppose that the linear functionals $\langle \underline{u}, \cdot \rangle$ and $\langle \underline{u}', \cdot \rangle$ agree. Why is it that $\underline{u} = \underline{u}'$?*

Under the assumption, we have $\langle \underline{u}, \underline{x} \rangle = \langle \underline{u}', \underline{x} \rangle$ for all $\underline{x} \in$V, that is $\langle \underline{u} - \underline{u}', \underline{x} \rangle = 0$. Now specifically for $\underline{x} = \underline{u} - \underline{u}'$ it follows that

$$\langle \underline{u} - \underline{u}', \underline{u} - \underline{u}' \rangle = 0\,,$$

and by the definition of an inner product this shows that $\underline{u} - \underline{u}' =$
zv, that is that $\underline{u} = \underline{u}'$.