# Math 121: Honours Integral Calculus

## Lecture 6

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# Last time



- $f : [a,b] \to \mathbb{R}$ bounded.
- $P : a = x_0 < x_1 < \cdots < x_n = b$ a paritition.
  - *Spacings* $\Delta x_i = x_i - x_{i-1}$.
  - *Mesh:* $\delta(P) = \max\{\Delta x_i\}_{i=1}^{n}$ (longest spacing).
- $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$, $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$.
- Riemann sums $L(f;P) = \sum_{i=1}^{n} m_i \Delta x_i$, $U(f;P) = \sum_{i=1}^{n} M_i \Delta x_i$.

## Definition

Say $f$ is integrable *on* $[a,b]$ *and that* $\int_a^b f(x)dx = I$ *if* $I \in \mathbb{R}$*is the* unique *number so that* $L(f;P) \le I \le U(f;P)$ *for all* $P$.

# Examples



- $f$ constant: $\int_a^b c\,dx = c(b-a)$.
- Dirichlet's function $D(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ is not integrable on any interval ($m_i = 0$, $M_i = 1$).
- What if $f(x) = \begin{cases} D(x) & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$ on $[0,1]$?
- What if $f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$?

# The choice of partition



## Theorem

*The following are equivalent:*

1. *$I$ is the unique number between the lower and upper sums;*
2. $I = \lim_{\delta(P)\to 0} L(f;P) = \lim_{\delta(P)\to 0} U(f;P)$*;*
3. *The sums for the uniform partition converge to $I$.*

Why non-uniform partitions?

## Example

$f(x) = \log x$ on $[a,b]$ $(a > 0)$. The natural partition is $x_i = a\left(\frac{b}{a}\right)^{i/n}$. After complicated calculation (see notes) get

$$\int_a^b \log dx = (b\log b - b) - (a\log a - a)\,.$$

# The interval

**Theorem**

*Let $f$ be Riemann integrable on $[a,b]$. Then $f$ is integrable on any sub-interval.*

**Theorem**

*Let $f$ be integrable on $[a,b]$, $[b,c]$. Then $f$ is integrable on $[a,c]$ and*

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx\,.$$

**Proof.**

Concatenate partitions. □

# The interval

## Definition

If $b < a$ set $\int_b^a f(x)dx = -\int_a^b f(x)dx$. If $a = b$ set the integral to zero.

## Example

$f(x) = \begin{cases} 5 & 2 \leq x < 3 \\ -2 & 3 < x \leq 5 \end{cases}$. Then
$\int_2^5 f(x)dx = \int_2^3 f(x)dx + \int_3^5 f(x)dx = 5 \cdot 1 + (-2) \cdot 2 = 1.$

## Example

We showed that $\int_0^b xdx = \frac{b^2}{2}$ (right triangle with both sides of length $b$). Since $\int_0^b = \int_0^a + \int_a^b$ it follows that
$\int_a^b xdx = \int_0^b xdx - \int_0^a xdx = \frac{b^2}{2} - \frac{a^2}{2}$.

# The function



**Theorem**

*Let $f, g$ be Riemann integrable on $[a, b]$. Let $A, B \in \mathbb{R}$. Then $Af + Bg$ is integrable on $[a, b]$ and*

$$\int_a^b (Af(x) + Bg(x))\,dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx\,.$$

**Example**

$\int_a^b (Ax + B)dx = A\int_a^b xdx + B\int_a^b 1dx = A\left(\frac{b^2}{2} - \frac{a^2}{2}\right) + B(b - a).$

# The function



**Theorem**

*Let $f$ be continous on $[a, b]$. Then $f$ is integrable on $[a, b]$.*

**Proof.**

Continuity: $f$ does not fluctuate much on small intervals. But this means that if $\delta(P)$ is small then $U(f; P) - L(f; P)$ is small (at most $b - a$ times the maximal fluctuation). It follows that there is at most one real number between all the lower and upper sums. □