Math 121 – Examples of partial fraction expansions

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1 $f(x) = \frac{x}{x^2-1}$

1. Find poles
   (a) Factor denominator: $x^2 - 1 = (x-1)(x+1)$.
   (b) Identify zeroes: ± 1.
2. Find asymptotics
   (a) $f(x) \sim_1 \frac{1}{(x-1)2} = \frac{1/2}{(x-1)}$
   (b) $f(x) \sim_{-1} \frac{-1}{(-2)(x+1)} = \frac{1/2}{x+1}$
3. Subtract: $f(x) - \left[\frac{1/2}{x-1} + \frac{1/2}{x+1}\right] = f(x) - \left[\frac{(x+1)+(x-1)}{2(x^2-1)}\right] = 0$.

Conclusion: $\frac{x}{x^2-1} = \frac{1/2}{x-1} + \frac{1/2}{x+1}$.

2 $f(x) = \frac{x}{x^4-2x^2+1}$.

1. Find poles
   (a) Factor denominator: $x^4 - 2x^2 + 1 = (x^2-1)^2 = (x-1)^2(x+1)^2$.
   (b) Identify zeroes: ± 1.
2. Find asymptotics
   (a) $f(x) \sim_1 \frac{1}{(x-1)^2 \cdot 4} = \frac{1/4}{(x-1)^2}$
   (b) $f(x) \sim_{-1} \frac{-1}{(-2)^2(x+1)^2} = \frac{-1/4}{(x+1)^2}$
3. Subtract: $f(x) - \left[\frac{1/4}{(x-1)^2} - \frac{1/4}{(x+1)^2}\right] = f(x) - \frac{(x+1)^2-(x-1)^2}{4(x-1)^2(x+1)^2} = 0$.

Conclusion: $\frac{x}{x^4-2x^2+1} = \frac{1/4}{(x-1)^2} - \frac{1/4}{(x+1)^2}$.

3 $f(x) = \frac{x^3}{x^4-2x^2+1}$.

1. Find poles
   (a) Factor denominator: $x^4 - 2x^2 + 1 = (x^2-1)^2 = (x-1)^2(x+1)^2$.
   (b) Identify zeroes: ± 1.
2. Find asymptotics
   (a) $f(x) \sim_1 \frac{1^3}{(x-1)^2 \cdot 4} = \frac{1/4}{(x-1)^2}$
   (b) $f(x) \sim_{-1} \frac{(-1)^3}{(-2)^2(x+1)^2} = \frac{-1/4}{(x+1)^2}$
3. Subtract:
   (a) $f(x) - \left[\frac{1/4}{(x-1)^2} - \frac{1/4}{(x+1)^2}\right] = \frac{x^3}{(x-1)^2(x+1)^2} - \frac{(x+1)^2-(x-1)^2}{4(x-1)^2(x+1)^2} = \frac{x^3-x}{(x-1)^2(x+1)^2}$.
   (b) New numerator *must have* factors corresponding to $(x-1)(x+1)$. Indeed, $x^3 - x = x(x-1)(x+1)$.
   (c) *Cancel factors*: $f(x) - \left[\frac{1/4}{(x-1)^2} - \frac{1/4}{(x+1)^2}\right] = \frac{x}{(x-1)(x+1)}$.
4. New asymptotics (note no new bad points!)
   (a) $\frac{x}{x^2-1} \sim_1 \frac{1}{(x-1)2} = \frac{1/2}{(x-1)}$
   (b) $\frac{x}{x^2-1} \sim_{-1} \frac{-1}{(-2)(x+1)} = \frac{1/2}{x+1}$

Conclusion: $\frac{x}{x^2-1} = \frac{1/2}{x-1} + \frac{1/2}{x+1}$ so $\frac{x^3}{x^4-2x^2+1} = \frac{1/4}{(x-1)^2} - \frac{1/4}{(x+1)^2} + \frac{1/2}{x-1} + \frac{1/2}{x+1} = \frac{1/4}{(x-1)^2} + \frac{1/2}{x-1} - \frac{1/4}{(x+1)^2} + \frac{1/2}{x+1}$.

4 $f(x) = \frac{1}{x^5+x^3}$.

1. Find poles
   (a) Factor denominator: $x^5 + x^3 = x^3(x^2+1)$.
   (b) Identify zeroes: 0.
2. Find asymptotics
   (a) $f(x) \sim_0 \frac{1}{x^3 \cdot 1} = \frac{1}{x^3}$.
3. Subtract:
   (a) $f(x) - \frac{1}{x^3} = \frac{x^2+1-1}{x^3(x^2+1)} = \frac{x^2}{x^3(x^2+1)}$.
   (b) New numerator *must have* factors corresponding to zero, in this case x^2.
   (c) *Cancel factors*: $f(x) - \frac{1}{x^3} = \frac{1}{x(x^2+1)}$.
4. New asymptotics (note no new bad points!)
   (a) $\frac{1}{x(x^2+1)} \sim_0 \frac{1}{x}$.
5. New subtraction
   (a) $\frac{1}{x(x^2+1)} - \frac{1}{x} = \frac{x^2+1-1}{x(x+1)^2} = \frac{x}{x^2+1}$.

Conclusion: $\frac{1}{x^5+x^3} = \frac{1}{x^3} + \frac{1}{x} + \frac{x}{x^2+1}$.