Math 121 – Pre-midterm sheet

Lior Silberman, UBC

February 2, 2012

1 About the exam

- In class on Wednesday, February 8th.
- 50 minutes: 14:00 - 14:50.
- No notes, calculators, etc.
- Material: everything up to absolute convergence of improper integrals.
- Study problems:
  - Past finals of 121 and 101 at `http://www.math.ubc.ca/Ugrad/pastExams/index.shtml`
  - Our Problem sets.
  - The textbook.

2 Summary of material

1. Summation nothations and formulas for sums.
2. The Riemann (definite) integral
   (a) Definition of Riemann sums
   (b) Explicit evaluation of Riemann sums using formulas as in (1)
   (c) The definition of the integral
   (d) Explicit limits of Riemann sums calculated as in (b).
   (e) Properties of integrals (concatenation of integrals,
   (f) The Fundamental Theorem of Calculus – evaluation of integral using anti-derivatives
3. Areas of plane regions bounded by graphs of functions
   (a) Express an area as a Riemann integral.
   (b) Evaluate the integral to calculate an area.
4. Techniques of integration – finding anti-derivatives
   (a) Integration by substitution (forward and backward)
   (b) Integration by parts
5. Improper integrals
   (a) Definition; evaluation using limits.
   (b) Using comparison and asymptotics to decide convergence wihtout evaluation.
   (c) Absolute convergence.

3 A few sample problems (not covering everything)

1. Let $F(X) = \int_{X^3(1+\cos X)}^{\infty} e^{-2x} \cos(x^2)\,dx$.

   (a) Show that the integral converges, so that $F(X)$ is well-defined for all X.

   (b) Show that $F(X)$ is differentiable as a function of X.

   (c) Find $\frac{dF}{dX}$.

2. Let $f(x) = x^3$.

   (a) Show that $\sum_{k=1}^{n} k^3 = \frac{k^2(k+1)^2}{4}$.

   (b) Let P_n be the partition of $[0,1]$ into the points $\left\{x_i = \frac{i}{n}\right\}_{i=0}^{n}$. Evaluate $L(f;P_n)$ and $U(f;P_n)$ as functions of n using (a).

   (c) Use (b) to show $f(x) = x^3$ is integrable on $[0,1]$ and to evaluate $\int_0^1 x^3\,dx$.

3. Evaluate the following integrals

   (a) $\int (x+1)\log x\,dx$

   (b) $\int_0^{\infty} \frac{dx}{(1+x)^3}$

4. Find $f(x)$ so that $f(x) = 1 + \int_0^x \frac{tf(t)}{1+t+t^2}\,dt$ (hint: $(\log f(x))' = \frac{f'(x)}{f(x)}$).

5. Let R be the finite region bounded above by $y = 4 - x^2$ and below by $y = 2 - x$. Find the area of this region.

(Solutions on the next page)

4 Solutions

1. Let $F(X) = \int_{X^3(1+\cos X)}^{\infty} e^{-2x}\cos(x^2)\,\mathrm{d}x$.

   (a) For all x we have $\left|e^{-2x}\cos(x^2)\right| \le e^{-2x}$ since $|\cos(y)| \le 1$ for all y. Since $\int_a^\infty e^{-2x}\,\mathrm{d}x$ converges, the integral defining F converges absolutely, and in particular is convergent.

   (b) By the addition formula for improper integrals we have $F(X) = \int_0^\infty e^{-2x}\cos(x^2) - \int_0^{X^3(1+\cos X)} e^{-2x}\cos(x^2)\,\mathrm{d}x$. Since the first term is a constant, it is enough to investigate the differentiability of the second term. Since $e^{-2x}\cos(x^2)$ is continuous, the Fundamental Theorem of Calculus gives that $\int_0^Y e^{-2x}\cos(x^2)\,\mathrm{d}x$ is differentiable with respect to Y. By the chain rule, since $Y = X^3(1+\cos X)$ is differentiable as a function of X, $F(X)$ is also differentiablew.

   (c) By the Fundamental Theorem of Calculus $\frac{d}{dY}\int_0^Y e^{-2x}\cos(x^2)\,\mathrm{d}x = e^{-2Y}\cos(Y^2)$ we may apply the chain rule to find:

$$\frac{dF}{dX} = 0 - \left(3X^2(1+\cos X) - X^3\sin X\right)e^{-2X^3(1+\cos X)}\cos\left(X^6(1+\cos X)^2\right).$$

2. Let $f(x) = x^3$.

   (a) We show this by induction on n. When $n=0$ the sum is empty and $\frac{0^2 1^2}{4} = 0$. Assume the claim holds for some n. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 && \text{(concatenation of sums)}\\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 && \text{(induction hypothesis)}\\
&= \frac{(n+1)^2}{4}\left[n^2+4(n+1)\right] = \frac{(n+1)^2}{4}\left[n^2+4n+4\right]\\
&= \frac{(n+1)^2(n+2)^2}{4} = \frac{(n+1)^2((n+1)+1)^2}{4}.
\end{aligned}$$

   In other words, the claim holds for $n+1$ as well. By induction the claim holds for all n.

   (b) The function x^3 is monotone increasing on $[0,1]$. It follows that the the minimum of f on any interval $[x_{i-1}, x_i]$ is attained at x_{i-1} and the maximum at x_i. It follows that

$$L(f;P_n) = \sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^3\frac{1}{n} = \frac{1}{n^4}\sum_{i=0}^{n-1} i^3 = \frac{(n-1)^2 n^2}{4n^4}$$

   and

$$U(f;P_n) = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\frac{1}{n} = \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4n^4}.$$

   (c) We have $L(f;P_n) = \frac{1}{4}\left(\frac{n-1}{n}\right)^2 = \frac{1}{4}\left(1-\frac{1}{n}\right)^2$ and $U(f;P_n) = \frac{1}{4}\left(\frac{n+1}{n}\right)^2 = \frac{1}{4}\left(1+\frac{1}{n}\right)^2$. Since $\lim_{n\to\infty}\left(1\pm\frac{1}{n}\right)^2 = 1$, we see that there are upper and lower Riemann sums arbitrarily close to $\frac{1}{4}$. It follows that $\frac{1}{4}$ is the unique number lying between all lower and upper sums, so $f(x)$ is integrable and $\int_0^1 x^3\,\mathrm{d}x = \frac{1}{4}$.

3. Evaluate the following integrals

   (a) We integrate by parts, differentiating $\log x$, to see: $\int (x+1)\log x\,\mathrm{d}x = \frac{(x+1)^2}{2}\log x - \int \frac{(x+1)^2}{2}\frac{1}{x}\,\mathrm{d}x = \frac{(x+1)^2}{2}\log x - \frac{1}{2}\int\left(x+2+\frac{1}{x}\right)\mathrm{d}x = \frac{(x+1)^2}{2}\log x - \frac{1}{2}\left(\frac{x^2}{2}+2x+\log x+C\right)$.

(b) Since $\frac{d}{dx}\left(-\frac{1}{2}(1+x)^{-2}\right) = \frac{1}{(1+x)^3}$ we have $\int_0^\infty \frac{\mathrm{d}x}{(1+x)^3} = \lim_{T\to\infty}\left[-\frac{1}{2(1+T)^2}+\frac{1}{2}\right] = \frac{1}{2}$.

4. Suppose f was a solution. differeniating the equation and using the fundamental theorem of calculus we find:

$$f'(x) = \frac{xf(x)}{1+x+x^2},$$

in other words that

$$(\log f)' = \frac{x}{1+x+x^2}.$$

Now $\int \frac{x\,\mathrm{d}x}{1+x+x^2} = \int \frac{(x+\frac{1}{2}-\frac{1}{2})\,\mathrm{d}x}{\frac{3}{4}+(x+\frac{1}{2})^2} = \frac{1}{2}\int \frac{(2x+1)\,\mathrm{d}x}{\frac{3}{4}+(x+\frac{1}{2})^2} - \frac{1}{2}\int \frac{\mathrm{d}x}{\frac{3}{4}+(x+\frac{1}{2})^2}$. In the first integral we set $u = 1+x+x^2$ so $du = (2x+1)dx$. In the second we set $v = 2\frac{x+\frac{1}{2}}{\sqrt{3}}$ so $dv = \frac{2}{\sqrt{3}}\,\mathrm{d}x$ to find:

$$\begin{aligned}
\int \frac{x\,\mathrm{d}x}{1+x+x^2} &= \frac{1}{2}\int\frac{\mathrm{d}u}{u} - \frac{\sqrt{3}}{4}\int \frac{\mathrm{d}v}{\frac{3}{4}+\left(\frac{\sqrt{3}}{2}v\right)^2} = \frac{1}{2}\log u - \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int\frac{\mathrm{d}v}{1+v^2} \\
&= \frac{1}{2}\log u - \frac{1}{\sqrt{3}}\arctan v + C \\
&= \frac{1}{2}\log(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.
\end{aligned}$$

Since $\log f$ has the same derivative as this function, it follows that for some constant C we have

$$f(x) = \exp\left\{\frac{1}{2}\log(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C\right\}.$$

To evaluate the constant note that in the given equation we must have $f(0) = 1+\int_0^0 \frac{tf(t)\,\mathrm{d}t}{1+t+t^2} = 1$ so $0 = \log f(0) = \frac{1}{2}\log 1 - \frac{1}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}} + C$. We conclude that $C = \frac{1}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}}$ so

$$f(x) = \exp\left\{\frac{1}{2}\log(1+x+x^2) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}}\right\}.$$

(Aside: since $\tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$ we have $C = \frac{\pi}{6\sqrt{3}}$).

5. We have $4-x^2 \geq 2-x$ precisely when $2 \geq x^2-x$, that is when $2\frac{1}{4} \geq \left(x-\frac{1}{2}\right)^2$, which is equivalent to $-\frac{3}{2} \leq x-\frac{1}{2} \leq \frac{3}{2}$. It follows that the area of R is

$$\begin{aligned}
\int_{-1}^{2}\left((4-x^2)-(2-x)\right)\mathrm{d}x &= \int_{-1}^{2}\left(2+x-x^2\right)\mathrm{d}x \\
&= \left[2x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{x=-1}^{2} \\
&= \left(4+2-\frac{8}{3}\right) - \left(-2+\frac{1}{2}+\frac{1}{3}\right) \\
&= 8-\frac{9}{3} = 5.
\end{aligned}$$