Math 121: Problem set 8 (due 13/3/12)

Practice problems (not for submission!)

Chapter 8: Problems on plotting and lengths. Problems on polar curves.
Sections 9.1: All problems.

Parametric curves

1. Give a simple description of the traces of the following curves:
   (a) $x(t) = \cos t + \sin t$, $y(t) = \cos t - \sin t$, $0 \leq t \leq 2\pi$.
   (b) $x(t) = \frac{1}{\cos t}$, $y(t) = \tan^2 t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2. Find the lengths of the following curves:
   SUPP $t \mapsto (e^t - t, e^t + t)$ for $a \leq t \leq b$
   Hint: One way to get rid of the square root is by substituting $u = \sqrt{e^{2t} + 1}$.
   (b) The spiral $r = \theta^2$ for $1 \leq \theta \leq 2$.
   (c) The spiral $r = \theta^{-2}$ for $\frac{1}{10} \leq \theta \leq 10$.

3. A curve passes through the point $(1,0)$ and has the slope $\frac{x}{e^y}$ at the point (x,y). What is the curve?
   Hint: The statement about the slope is a differential equation.

4. Let $L > 0$ be a parameter. A curve $y = y(x)$ in the quadrant $x, y \geq 0$ begins at the point $(x_0, 0)$ where $x_0 > L$, and has the property that for each point (x,y) on the curve, the segment of the tangent line at (x,y) that stretches between that point and the y-axis has length $\frac{x^2}{L}$.
   (a) Express the condition as an equation involving $\frac{dy}{dx}$, x and L.
   (b) Find $y(x)$.
   SUPP What happens if $x_0 = L$?

Limits

5. *Bernoulli's inequality* and an application.
   (a) Show that for every real $x \geq -1$ and every natural number n the inequality $(1+x)^n \geq 1 + nx$ holds.
   (b) Fix $a > 1$ and suppose that $a^{1/n} > 1 + \varepsilon$ for some $\varepsilon > 0$. Show that $n < \frac{a-1}{\varepsilon}$.
   (c) Conclude that $\lim_{n\to\infty} \sqrt[n]{a} = 1$.
   (d) Let $0 < b \leq 1$. Show that $\lim_{n\to\infty} b^{1/n} = 1$.

6. In each case either show that the limit exists and evaluate it or show that it does not exist.
   (a) $\lim_{n\to\infty} \frac{n^2 - 5n + \cos n}{1 + \sqrt{n} + 3n^2}$
   (b) $\lim_{n\to\infty} \frac{(2n)!}{(n!)^2}$
   Hint: Compare a_{n+1} to a_n.
   (c) $\lim_{n\to\infty} \left(\sqrt{n^2 + 7n} - n\right)$.

Supplementary problem: Reparametrization of curves

A. Let $\gamma(t) = (x(t), y(t))$ be a differentiable curve defined on the interval $[a,b]$ (this means that $x(t), y(t)$ are differentiable functions of t). Let $f\colon [c,d] \to \mathbb{R}$ be a differentiable function such that $f(c) = a$, $f(d) = b$ and $f'(x) > 0$ for all $c < x < d$. Let $\tilde{\gamma}(s)$ be the curve $\tilde{\gamma}(s) = \gamma(f(s)) = (x(f(s)), y(f(s)))$.
   (a) Show that the range of f is exactly $[a,b]$ and that every point of a,b is of the form $f(x)$ for a unique $x \in [c,d]$.
   (b) Show that γ and $\tilde{\gamma}$ have the same length.

Supplementary problem: Newton's Method

B. Let f be twice differentiable on $[a,b]$. Suppose that for some $x_0 \in (a,b)$ we have $f(x_0) = 0$ and $f'(x_0) \neq 0$, and define an auxilliary function $G(x) = x - \frac{f(x)}{f'(x)}$, at least if $f'(x) \neq 0$.
   (a) Let I be an interval where f' does not vanish. Show that for all x, y in that interval there is ξ between them so that $G(x) - G(y) = \frac{f(\xi)f''(\xi)}{(f'(\xi))^2}(x-y)$.
   (b) Show that there is $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subset [a,b]$ and so that if $|\xi - x_0| < \delta$ then $f'(\xi) \neq 0$ and $\left|\frac{f(\xi)f''(\xi)}{(f'(\xi))^2}\right| \leq \frac{1}{2}$.
   (c) Use (a),(b) to show that if $x \in (x_0 - \delta, x_0 + \delta)$ then $G(x)$ belongs to the same interval.
   (d) Choose $a_0 \in (x_0 - \delta, x_0 + \delta)$ and define a sequence by $a_{n+1} = G(a_n)$. Show that this is well-defined and that $|a_{n+1} - x_0| \leq \frac{1}{2}|a_n - x_0|$.
   (e) Conclude that $|a_n - x_0| \leq \frac{1}{2^n}|a_0 - x_0|$ and hence that $\lim_{n\to\infty} a_n = x_0$.
   RMK You have shown that Newton's method will actually find a non-degenerate root if started close enough to it.

Supplementary problems: Topology of the real line

C. Call a subset $U \subset \mathbb{R}$ *open* if for every $x \in U$ there is $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset U$.
   (a) Show that (a,b) is open if $a < b$.
   (b) Show that $\left\{x \mid 2 < e^{x^2} < 3\right\}$ is open.
   (c) Let f be a real-valued function defined on a subset of $\mathbb{R}$. Show that f is continuous if and only if for every open subset $U \subset \mathbb{R}$ the preimage $f^{-1}(U) \stackrel{\text{def}}{=} \{x \mid f(x) \in U\}$ is open.
   Hint: Suppose f is continuous and that $f(x) \in U$. Use the definition of continuity to show that there is an interval about x which is mapped by f into U. Now show the converse.
   (d) Show that the class of open sets includes the empty set, all of $\mathbb{R}$.
   (e) Show that if U, V are open then so are $U \cap V$ and $U \cup V$.
   (f) Show that if $\mathcal{U}$ is a collection of open subset of $\mathbb{R}$ then its union $\bigcup \mathcal{U}$ is also open.

D. Call a subset $A \subset \mathbb{R}$ *closed* if its complement $\mathbb{R} \setminus A = \{x \in \mathbb{R} \mid x \notin A\}$ is open.
   (a) Suppose A is closed and let $\{a_n\}_{n \geq n_0} \subset A$ converge to $L \in \mathbb{R}$. $L \in A$.
   Hint: Suppose $\lim_{n\to\infty} a_n$ belonged to the open set $\mathbb{R} \setminus C$ ...
   (b) Suppose A has the property that it contains the limit of every convergent sequence whose elements are in A. Show that A is closed.
   Hint: Let $x \in \mathbb{R}$. Show that if for every n there is $a_n \in A \cap \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ then $x \in A$ and conclude that if $x \notin A$ then a full interval about x is outside A.
   (c) Show that f is continuous if and only if $f^{-1}(A)$ is closed for every closed $A \subset \mathbb{R}$.

E. Show a function f defined on a set $A \subset \mathbb{R}$ is continuous if and only if for every sequence $\{x_n\}_{n \geq n_0} \subset A$ which converges to a limit $L \in A$ one has $\lim_{n\to\infty} f(x_n) = f(L)$.