Math 613: Problem set 6 (due xx/11/09)

Dirichlet characters

Let $N \geq 1$. A *Dirichlet character mod N* is a non-zero function $\chi\colon \mathbb{Z} \to \mathbb{C}$ such that $\chi(ab) = \chi(a)\chi(b)$, $\chi(a) = \chi(b)$ if $a \equiv b\,(N)$ and $\chi(a) = 0$ whenever $(a,N) > 1$. We freely identify χ with the function it induces on $\mathbb{Z}/N\mathbb{Z}$

1. Let χ be a Dirichlet character mod N.
   (a) Show that $\chi(1) = 1$ and that χ is non-zero in $(\mathbb{Z}/N\mathbb{Z})^{\times}$.
   (b) Show that the non-zero values taken by χ are roots of unity.
   (c) Let ψ, χ be two Dirichlet characters mod N. Show that $\psi\chi$ is also such a character.

2. (The conductor) Let χ be a Dirichlet character mod N.
   (a) Let $N|M$. Show that the function $\chi_M(a) = \begin{cases} \chi(a) & (a,M) = 1 \\ 0 & (a,M) > 1 \end{cases}$ is a Dirichlet character mod M satisfying $\chi_M(1+kN) = 1$ if $(1+kN, M) = 1$.
   — Characters mod M obtained this way with $N < M$ are called *imprimitive*. Other characters are called *primitive*.
   (b) Assume that $\chi_M = \psi_M$ for another character ψ mod N. Show that $\chi = \psi$.
   (c) Show that χ is primitive iff $\bar{\chi}$ is.
   (d) Let $Q_\chi = \{q \geq 1 \mid \forall n \in \mathbb{Z}\colon\ (n \equiv 1\,(q) \wedge (n,N) = 1) \Rightarrow \chi(n) = 1\}$. Show that if $q_1, q_2 \in Q_\chi$ then $(q_1, q_2) \in Q_\chi$.
   (e) Let $m_\chi = \min Q_\chi$. Show that there is a primitive character ψ mod m_χ such that $\chi = \psi_N$.

DEFINITION. Call m_χ the *conductor* of χ.

The Gauss Sum

3. (Fourier analysis on $\mathbb{Z}/N\mathbb{Z}$) Write $L^2(\mathbb{Z}/N\mathbb{Z})$ for the space of $\mathbb{C}$-valued functions on $\mathbb{Z}/N\mathbb{Z}$, equipped with the norm
$$\|f\|_{L^2}^2 = \sum_{a\,(N)} |f(a)|^2 .$$
   For $f \in L^2(\mathbb{Z}/N\mathbb{Z})$ and $k \in \mathbb{Z}/N\mathbb{Z}$ set $\hat{f}(k) = \sum_{a(N)} f(a)e(-\frac{ak}{N})$.
   (a) For $a \in \mathbb{Z}/N\mathbb{Z}$ show that $\sum_{k\in\mathbb{Z}/N\mathbb{Z}} e(\frac{ak}{N}) = N\delta_{a,0}$ (Kronecker delta).
   (b) (Parseval formula) Show that $\left\|\hat{f}\right\|_{L^2} = \|f\|_{L^2}$.
   (c) (Fourier inversion) Show that $f(a) = \frac{1}{N}\sum_{k\,(N)} \hat{f}(k)e(\frac{ka}{N})$.

4. (The Gauss sum) Let χ be a primitive Dirichlet character mod N. For $b \in \mathbb{Z}/N\mathbb{Z}$ we define the *Gauss sum*
$$G(\chi, b) = \sum_{a\,(N)} \chi(a)e(\frac{ab}{N}) .$$
   In particular, write $G(\chi) = G(\chi, 1)$.

(a) Let $(c,N)=1$. Show that $G(\chi,bc)=G(\chi,b)\bar{\chi}(c)$. Conclude that if $(b,N)=1$ we have $G(\chi,b)=G(\chi)\cdot\bar{\chi}(b)$.
(b) Let $(b,N)>1$. Show that $G(\chi,b)=0$.
(c) Evaluate $|G(\chi)|$.
Hint: Use Parseval's identity.
(d) Show that $\chi(a)=\frac{G(\chi)}{N}\sum_{b\,(N)}\bar{\chi}(b)e(-\frac{ab}{N})$.
(e) Show that $G(\bar{\chi})=\chi(-1)\overline{G(\chi)}$.

5. (Twisted Poisson summation) Fix a primitive Dirichlet character χ of conductor q.
(a) For $f\in\mathcal{S}(\mathbb{R}^n)$ set $(D_a f)(x)=f(ax)$, and $(M_p f)(x)=f(x)e(-px)$. Show that $\widehat{D_a f}(k)=\frac{1}{a^n}\hat{f}(\frac{k}{a})$ and $\widehat{M_p f}(k)=\hat{f}(k+p)$.
(b) Prove the *twisted Poisson summation formula*: for $f\in\mathcal{S}(\mathbb{R})$ one has
$$\sum_{n\in\mathbb{Z}}f(n)\chi(n)=\frac{G(\chi)}{q}\sum_{k\in\mathbb{Z}}\hat{f}(\frac{k}{q})\bar{\chi}(k).$$
Hint: use the formula from 4(d).

Dirichlet L-functions

Fix a primitive Dirichlet character χ of conductor q. We assume that $q>1$, so in particular $\chi(0)=0$. For $\varphi\in$

6. (cf PS 2, problem 8) For $\varphi\in\mathcal{S}(\mathbb{R})$ and $r>0$ set $\varphi_\chi(r)=\sum_{n\in\mathbb{Z}}\varphi(rn)\chi(n)$ and $Z(\chi;\varphi;s)=\int_0^\infty\varphi_\chi(q^{-1/2}r)r^s\frac{dr}{r}$.
(a) Show that the sum converges absolutely.
(b) Show that as $r\to\infty$, $\left|\varphi_\chi(r)\right|$ decays faster than any polynomial and that as $r\to 0$, $\left|\varphi_\chi(r)\right|=O(r^{-1})$. Conclude that $Z(\chi;\varphi;s)$ converges absolutely in $\Re(s)>1$ and defines a holomorphic function there.
(c) Show that for $\Re(s)>1$,
$$Z(\chi;\varphi;s)=\int_1^\infty\varphi_\chi\left(\frac{r}{\sqrt{q}}\right)r^s\frac{dr}{r}+\frac{G(\chi)}{\sqrt{q}}\int_1^\infty\hat{\varphi}_{\bar{\chi}}\left(\frac{r}{\sqrt{q}}\right)r^{1-s}\frac{dr}{r}$$
(d) Since $\varphi\in\mathcal{S}(\mathbb{R})$ we also have $\hat{\varphi}\in\mathcal{S}(\mathbb{R})$ and $\hat{\hat{\varphi}}(x)=\varphi(-x)$. Conclude that $Z(\chi;\varphi;s)$ extends to an entire function of s which satisfies the *functional equation*
$$Z(\chi;\varphi;s)=\frac{G(\chi)}{\sqrt{q}}Z(\bar{\chi};\hat{\varphi};1-s).$$
(e) Assume that φ has the same parity as χ, that is that $\varphi(-x)=\chi(-1)\varphi(x)$, and show that for $\Re(s)>1$ we have
$$Z(\chi;\varphi;s)=2q^{s/2}\left(\int_0^\infty\varphi(r)r^s\frac{dr}{r}\right)L(s;\chi).$$
(f) For $\varphi\in C_c^\infty(\mathbb{R}^\times)$, $\left(\int_0^\infty\varphi(r)r^{ns}\frac{dr}{r}\right)$ extends to an entire function; conclude that $E(\Lambda;s)$ extends to a meromorphic function of s.

(g) Set $a \in \{0,1\}$ so that $\chi(-1) = (-1)^a$. For $\varphi(x) = x^a \exp\left\{-\pi x^2\right\}$ show that $\hat{\varphi}(x) = (-i)^a \varphi(x)$ and that

$$\int_0^\infty \varphi(r) r^s \frac{dr}{r} = 2\Gamma_{\mathbb{R}}(s+a)$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. Conclude that $L(s;\chi)$ is an entire function, and that the *completed L-function* $\Lambda(s;\chi) = q_\chi^{s/2}\Gamma_{\mathbb{R}}(s+a)L(s;\chi)$ satisfies the functional equation

$$\Lambda(s;\chi) = \varepsilon(\chi)\Lambda(1-s;\bar{\chi}),$$

where $\varepsilon(\chi) = q^{-1/2}G(\chi)(-i)^a$. Show that $|\varepsilon(\chi)| = 1$.

Boundedness in vertical strips

OPT (The Phragmen-Lindelöf Theorem) Let $f(s)$ be continuous in the strip $S = \{\sigma + it \mid \sigma \in [a,b]\}$ and holomorphic in the interior of the strip.

(a) (Simple version) Assume that $|f(s)| \le M$ on ∂S and that f is bounded on S. Then $|f(s)| \le M$ on S.
Hint: Apply the maximum principle to $f(s) \cdot e^{\varepsilon(s-s_0)^2}$ in an appropriate domain.

— (Functions of finite order) For the rest of the problem we assume that for some $C, A \ge 0$ and all $s \in S$ we have the a-priori bound$|f(s)| \le Ce^{|s|^A}$.

(b) Show $|f(s)| \le M$ on ∂S implies $|f(s)| \le M$ on S.

(c) (Interpolation) Assume that $|f(a+it)| \le M_a$ and that $|f(b+it)| \le M_b$. Let $\tau(\sigma)$ be the linear function such that $\tau(a) = 0$, $\tau(b) = 1$. Show that for $a \le \sigma \le b$ we have

$$|f(\sigma + it)| \le M_a^{1-\tau} M_b^{\tau}.$$

Hint: Find a function g such that $|g(s)|$ is precisely the RHS.

(d) (Polynomial growth on the boundary) Assume that $|f(a+it)| \le M(1+|t|)^\alpha$ and that $|f(b+it)| \le M(1+|t|)^\beta$. Show that for some $M' \ge 0$ and all $a \le \sigma \le b$ we have

$$|f(\sigma+it)| \le M'(1+|t|)^{(1-\tau(\sigma))\alpha + \tau(\sigma)\beta}.$$

(e) Assume now that f is meromorphic in the strip, with only finitely many poles, and that the assumptions of (d) hold away from fixed neighbourhoods of the poles. Show that, away from these neighbourhoods, the conclusion holds up to a loss in the constant.

8. (Boundedness in vertical strips) Let $D(s) = \sum_{n\ge 1} a_n n^{-s}$ be a Dirichlet series which converges absolutely in $\Re(s) > 1$, and write $\bar{D}(s) = \sum_{n \ge 1} \overline{a_n} n^{-s}$. Assume that we have a Γ-factor

$$\gamma(s) = \prod_{j=1}^d \Gamma_{\mathbb{R}}(s+\kappa_j)$$

with $\Re(\kappa_j) > -1$ and a conductor q so that $\Lambda(s) = q^{s/2}\gamma(s)D(s)$ is a function of finite order with finitely many poles in the critical strip and satisfies the fuctional equation

$$\Lambda(s) = \varepsilon\bar{\Lambda}(1-s)$$

with $\bar{\Lambda}(s) = q^{s/2}\gamma(s)\bar{D}(s)$. Now that the Γ-factor has no zeroes and by the assumption on the κ_j it has no poles in $\Re(s) \ge 1$.

(a) Use the functional equation to show that $\Lambda(s)$ has no poles in $\Re(s) < 0$.