# Math 100 §105, Fall Term 2010
# Sample Midterm Exam

November $8^{th}$,2010

| Student number: |
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| LAST name: |
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| First name: |
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## Instructions

- Do not turn this page over until instructed. You will have 45 minutes for the exam.
- You may not use books, notes or electronic devices of any kind.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.

| Signature: |
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| 1 | | /18 |
| 2 | | /8 |
| 3 | | /4 |
| 4 | | /10 |
| Total | | /40 |

# 1 Short-form answers

Show your work and clearly delineate your final answer. Not all problems are of equal difficulty.

**[3] a. If $x^2y^2 + x\sin y = 4$, find $\frac{dy}{dx}$.** Differentating with respect to $x$ and using the chain rule, we have:

$$2xy^2 + 2x^2y \cdot y' + \sin y + x\cos y \cdot y' = 0\,.$$

Solving for $y'$ we find

$$y' = -\frac{2xy^2 + \sin y}{2x^2y + x\cos y}\,.$$

**[3] b. Let $f(x) = x^3 \ln x$. Find the $f^{(4)}(x)$, the fourth derivative of $f$.** $f'(x) = 3x^2\ln x + x^2$; $f''(x) = 6x\ln x + 5x$; $f^{(3)}(x) = 6\ln x + 11$;

$$f^{(4)}(x) = \frac{6}{x}\,.$$

**[3] c. Differentiate $(\tan x)^x$.** We use the logarithmic differentiation formula $f' = f \cdot (\ln f)'$. In this case this gives:

$$\begin{aligned}((\tan x)^x)' &= (\tan x)^x\,(x\ln\tan x)' \\ &= (\tan x)^x\left(\ln\tan x + \frac{x}{\tan x}(\tan x)'\right) \\ &= (\tan x)^x\left(\ln\tan x + \frac{x}{\tan x}(1+\tan^2 x)\right)\,.\end{aligned}$$

**[3] d. Write down the first three nonzero terms in the Maclaurin series for $x\sin(-2x)$.** We have $\sin(u) \approx u - \frac{1}{6}u^3 + \frac{1}{120}u^5$ to fifth order. It follows that $\sin(-2x) \approx -2x + \frac{8}{6}x^3 - \frac{32}{120}x^5$ to fifth order. Multiplying by $x$ we find:

$$x\sin(-2x) \approx -2x^2 + \frac{4}{3}x^4 - \frac{4}{15}x^6$$

to 6th order.

**[3] e. Use a linear approximation to approximate $\sqrt{100.2}$.** Let $f(x) = \sqrt{100+x}$ where $f'(x) = \frac{1}{2\sqrt{100+x}}$. Then $f(0) = 10$, $f'(0) = \frac{1}{20}$ so

$$f(0.2) \approx 10 + \frac{0.2}{20} = 10.01\,.$$

**[3] f. Give an upper bound for the magnitude fo the error in your answer to part e.** We have $f''(x) = -\frac{1}{4\sqrt{100+x}^3}$. This decreases in magnitue with $x$, so for $0 \le c$ we have $|f''(c)| \le |f''(0)| = \frac{1}{4000}$. By the Lagrange form of the remainder in Taylor's Theorem the error in the approximation is at most

$$\frac{1}{2\cdot 4000}(0.2)^2 = \frac{4}{8\cdot 10^5} = 5\cdot 10^{-6}\,.$$

## 2 Long-form answers

**[4] The normal temperature of your Vancouver apartment is 23 degrees; the daytime temperature outside is about 5 degrees. A window remains open when you leave for UBC at 7am. By 1pm, the temperature in the apartment has dropped to 11 degrees. What will the temperature be at 7pm?** We assume that the temperature in the apartment will approach the outside temperature exponentially. In 6 hours the difference in temperatures dropped by a factor of 3 (from 18 degrees to 6). In 6 more hours we then expect a drop by the same factor, so the difference in temperature should then be 2 degrees, giving a temperature of $5+2=7$ degrees at 7pm.

## 3 Long-form answers

**[8] A trough is 10 m long and its ends have the shape of equilateral triangles (i.e. all three sides have equal length) that are 2 m across, situated with their points down. If the trough is being filled with water at the rate of $12\mathrm{m}^3/\mathrm{min}$, how fast is the water level rising when the water is $60$cm deep?** When the water level has reading height $h$ the water occupies a volume of length $L = 10\mathrm{m}$ each of whose cross-sections is an equilateral triangle of altitude $h$ (since the cross-sectional triangle is similar to the triangular cross-section of the trough). An equilateral triangle of height $h$ has sides $2h\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}h$. Its area is therefore $\frac{1}{2}(\frac{2}{\sqrt{3}}h)h = 3^{-1/2}h^2$. The volume of the water in the trough at time $t$ is then

$$V(t) = \frac{1}{\sqrt{3}}Lh(t)^2 .$$

Differentiating we find by the chain rule that

$$\frac{dV}{dt} = \frac{2}{\sqrt{3}}Lh(t)\frac{dh}{dt} .$$

We are given that $\frac{dV}{dt} = 12\frac{\mathrm{m}^3}{\mathrm{min}}$. Solving for $h'$ we find

$$h' = \frac{\sqrt{3}\cdot 12}{2\cdot 10\cdot h(t)}\frac{\mathrm{m}}{\mathrm{min}}$$

if $h(t)$ is given in metres. When $h = 0.6\mathrm{m}$ we then have

$$h' = \frac{\sqrt{3}\cdot 6}{10\cdot 0.6}\frac{\mathrm{m}}{\mathrm{min}} = \sqrt{3}\frac{\mathrm{m}}{\mathrm{min}} .$$

## 4 Long-form answers

Consider the function $f(x) = \sqrt{1 - xe^{-x/a}}$ on the interval $[0, 1]$. Here $a$ is a positive parameter

**[5] a. Find the absolute maximum of $f$ on the interval.** For $a > 0$ and $x \geq 0$ we have $-x/a \leq 0$ so $0 \leq e^{-x/a} \leq 1$. It follows that $xe^{-x/a} \leq 1$ so $1 - xe^{-x/a} \geq 0$ and $f$ is defined and continuous on $[0, 1]$. Also, for $x \geq 0$ $xe^{-x/a} \geq 0$ so $1 - xe^{-x/a} \leq 1$. It follows that $f(x) \leq \sqrt{1} = f(0)$ so the absolute maximum of $f$ is 1, and it occurs at $x = 0$.

**[5] a'. Find the absolute minimum of $f$ on the interval.** Since the square-root function is monotone its domain, it's enough to minimize $g(x) = 1 - xe^{-x/a}$. This function is clearly differentiable everywhere, so its absolute maximum on $[0, 1]$ exists and occurs either at an endpoint or at a critical point. We record:

$$g(0) = 1, \qquad g(1) = 1 - e^{-1/a} < 1$$

We have $g'(x) = -e^{-x/a} + \frac{x}{a}e^{-x/a} = \left(\frac{x}{a} - 1\right)e^{-x/a}$. Since $e^{-x/a}$ never vanishes, the only possible critical point is at $x_0 = a$. If $a > 1$ this is outside the interval, and the absolute minimum is $\sqrt{1 - e^{-1/a}}$ at $x = 1$. Otherwise we note that $g' < 0$ for $x < a$ and $g' > 0$ for $x > a$. It follows that $g$ is increasing between $x_0 = a$ and 1 so $g(a) < g(1)$, and the absolute minimum of $f$ is, in that case, $f(a) = \sqrt{1 - ae^{-a/a}} = \sqrt{1 - \frac{a}{e}}$.

**[2] b. Let $F(a)$ be your answer to part a. Assuming that $a$ is very small, write down a linear approximation to $F(a)$.** We have $F(a) = 1$ so this is also the linear approximation.

**[2] b'. Let $F(a)$ be your answer to part a'. Assuming that $a$ is very small, write down a linear approximation to $F(a)$.** For $0 < a < 1$ we have $F(a) = \sqrt{1 - \frac{a}{e}}$. We note first that $\lim_{a \to 0} F(a)$ exists an equals 1, so we define $F(0) = 1$. The resulting function is differentiable at 0, and its derivative is $F'(0) = \left[\frac{1}{2\sqrt{1 - \frac{a}{e}}} \cdot \left(-\frac{1}{e}\right)\right]_{a=0} = -\frac{1}{2e}$ so the linear approximation is

$$F(a) \approx 1 - \frac{a}{2e}\,.$$

**[3] c. Find the absolute minimum and maximum of $f(x) = e^{-|x|}$ on the interval $[-10, 10]$. Where are they attained?** Since $f(x) = f(-x)$ it's enough to study $f$ on the interval $[0, 10]$ where $f(x) = e^{-x}$. This function is continuous and differentiable there, and has no critical points (for $x > 0$ we have $f'(x) = e^{x}$ and this never vanishes) so its exterma occur at the endpoints of that interval. $f(0) = 1$ and $f(10) = e^{-10} < 1$. It follows that the absolute maximum of $f$ on $[-10, 10]$ is 1, attained at 0 and that the absolute minimum is $e^{-10}$, attained at $\pm 10$.