Math 422/501: Problem set 3 (due 30/9/09)

Groups of small order

1. Let m be a positive integer. Let C_m be the cyclic group of order m. Show that $\mathrm{Aut}(C_m) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$.
   Hint: Fix a generator g of C_m, and given $\varphi \in \mathrm{Aut}(C_m)$ consider $\varphi(g)$.

2. (Quals September 2008) Show that every group of order 765 is Abelian.
   Hint: To start with, let G act by conjugation on a normal Sylow p-subgroup.

3. Let G be a group of order 36 and assume that it does not have a normal Sylow 3-subgroup. Obtain a non-trivial homomorphism $G \to S_4$ and conclude that G is not simple.

Index calculations

4. Let G be a group, $H < G$ a subgroup of finite index. Show that there exists a normal subgroup $N \lhd G$ of finite index such that $N \subset H$.
   Hint: You can get inspiration from problem 3.

5. (Normal p-subgroups)
   (a) Let G be a finite group, $N \lhd G$ a normal subgroup which is a p-group. Use the conjugacy of Sylow subgroups to show that N is contained in every Sylow p-subgroup of G.
   (b) Now let G be any group, $N \lhd G$ a normal subgroup which is a p-group. Let $P < G$ be another p-subgroup. Show that PN is a p-subgroup of G and conclude that N is contained in every Sylow p-subgroup of G.

Commutators

Let G be a group. For $x, y \in G$ write $[x,y] = xyx^{-1}y^{-1}$ for the *commutator* of x, y. Write G' for the subgroup of G generated by all commutators and call it the *derived subgroup* of G.

6. (The abelianization)
   (a) Show that $x, y \in G$ commute iff $[x,y] = e$.
   (b) Show that G' is a normal subgroup of G.
   Hint: Show that it is enough to show that the set of commutators is invariant under conjugation. Then show that $g[x,y]g^{-1}$ is a commutator.
   (c) Show that $G^{\mathrm{ab}} = G/G'$ is abelian.
   (d) Let A be an Abelian group, and let $f \in \mathrm{Hom}(G,A)$. Show that $G' \subset \mathrm{Ker} f$. Conclude that f can be written uniquely as the composition of the quotient map $G \twoheadrightarrow G^{\mathrm{ab}}$ and a map $f^{\mathrm{ab}} \colon G^{\mathrm{ab}} \to A$.
   OPTIONAL Let G, H be groups and let $f \in \mathrm{Hom}(G,H)$. Does f extend to a map $G^{\mathrm{ab}} \to H^{\mathrm{ab}}$?

7. (Groups of Nilpotence degree 2) Let G be group, $Z = Z(G)$ its center.
   (a) Show that the commutator $[x, y]$ only depends on the classes of x, y in $G/Z(G)$.
   From now on assume that G is non-Abelian but that $A = G/Z$ is.
   (b) Show that $G' < Z(G)$.
   Hint: 6(d).
   (c) Show that the commutator map of G descends to an anti-symmetric bilinear pairing $[\cdot, \cdot] : A \times A \to Z(G)$.

8. Let G be a non-abelian group of order p^3.
   (a) Show that $Z(G) < G'$.
   Hint: 6(b) and general properties of p-groups.
   (b) Show that $Z(G) = G'$.
   Hint: Show that $G/Z(G)$ is abelian and use 7(b).

Optional: Example of a Sylow subgroup

A. Let k be field, V a vector space over k of dimension n. A *maximal flag* F in V is a sequence $\{0\} = F_0 \subsetneq F_1 \subseteq \cdots \subsetneq F_n = V$ of subspaces. Let $\mathscr{F}(V)$ denote the space of maximal flags in V. An ordered basis $\left\{\underline{v}_j\right\}_{j=1}^{n} \subset V$ is said to be *adapted* to F if $F_k = \mathrm{Sp}\left\{\underline{v}_j\right\}_{j=1}^{k}$ for all $0 \leq k \leq n$.
   (a) Show that the group $\mathrm{GL}(V)$ of all invertible k-linear maps $V \to V$ acts transitively on $\mathscr{F}(V)$.
   (b) Let $F \in \mathscr{F}(V)$ and let $B < \mathrm{GL}(V)$ be its stabilizer. Let $N = \{b \in B \mid \forall k \geq 1 \forall \underline{v} \in F_k : g\underline{v} - \underline{v} \in F_{k-1}\}$. Show that N is a normal subgroup of B.
   (c) Show that $B/N \simeq (k^\times)^n$.

B. Assume $|k| = q = p^r$ for a prime p. Let $V = k^n$, Let $G = \mathrm{GL}(V) = \mathrm{GL}_n(F)$, and let $B \subset G$ be the point stabilizer of the *standard flag* $V_k = \mathrm{Sp}\left\{\underline{e}_j\right\}_{j=1}^{k}$ where $\underline{e}_j$ is the jth vector of the standard basis.
   (a) What is $|\mathscr{F}(F^n)|$?
   Hint: For each one-dimensional subspace $W \subset V$ show that the set flags containing W is in bijection with the set flags $\mathscr{F}(V/W)$.
   (b) Show that q is relatively prime to $|\mathscr{F}(V)|$. Conclude that B contains a Sylow p-subgroup of G.
   (c) Show that N is a Sylow p-subgroup of G.

Optional: Infinite Sylow Theory

C. Let G be any group, $P < G$ a p-subgroup of finite index. We will show that Sylow's Theorems apply in this setting.
   (a) Show that G has a normal p-subgroup N of finite index.
   (b) Show that every Sylow p-subgroup contains N.
   (c) Deduce a version of Sylow's Theorem for G from Sylow's Theorems for G/N.