Math 342 Problem set 2 (due 19/1/09)

The natural numbers

1. Prove parts (1),(2),(3) of Proposition 2.1.3 on page 9 of the lecture notes.
   Hint: One half of (3) was done in class without induction. For the rest, in each case you need to choose which variable you want to use for your induction.

Division with remainder

2. (Parity)
   (a) Show that every integer m is of one of the forms $2n$, $2n+1$ for another integer n by quoting the Division Theorem. We call integers of the first form *even*, of the second *odd*, and call this proprety *parity* (for example, you can say that 5 has *odd parity*).
   (b) What is the parity of 10? 17? -9?
   (c) Show that the parity of the sum of two integers only depends on the parity of these integers, not on their values. Make an addition table for parities and compare it with the addition table of $\mathbb{F}_2$.

   [You should try to figure out for yourself how to use the usual proprties of addition in $\mathbb{Z}$ such associativity and commutativity to deduce the corresponding properties in $\mathbb{F}_2$.]

3. (§3A.E1, again)
   (a) What are the possible remainders when dividing an integer by 3?
   (b) By writing an arbitrary integer m in the form $3n+r$, show that one of $m, m+1, m+2$ is divisible by 3.
   Hint: divide into cases depending on the value of r.
   (c) Is this solution fundamentally different from the one given in Problem Set 1? In other words, where did we use induction to get this solution?

The Fibonacci sequence

Let $\{a_n\}_{n=0}^{\infty}$ be the sequence of integers defined as follows: $a_0 = 0$, $a_1 = 1$, and for $n \geq 1$, $a_{n+1} = a_n + a_{n-1}$. (Story: at month 1 we introduce a pair of newborn rabbits into the country; every pair of rabbits takes one month to mature, after which they spawn another pair every month; thus the number of pairs of rabbits at any month equals the number of pairs the pervious month, plus one new pair for each pair that was alive the month before).

4. Calculate a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 and check that $a_{10} = 55$.

5. For $1 \leq n \leq 10$ calculate the ratio $\frac{a_n}{a_{n-1}}$ to three decimal digits.

6. Let $R > 1$ be a solution of $x^2 - x - 1 = 0$. Calculate R to three decimal digits.

7. (§3D.E3(i)) Show that $a_n = \frac{R^n - r^n}{R - r}$ for all n, where r is the other solution to the equation.
   Hint: Check the cases $n = 0$, $n = 1$ by hand and use induction.

REMARK. We will return to this sequence in the next problem set.