Gaps of saddle connection directions for some branched covers of tori

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West Coast Dynamics Seminar

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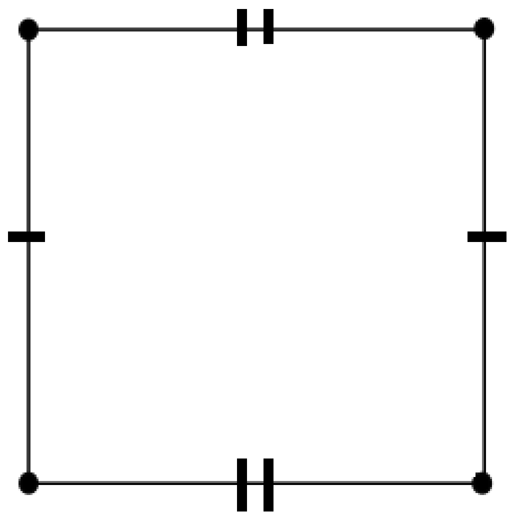
Translation surfaces

A **translation surface** is a collection of polygons with edge identifications given by translations.



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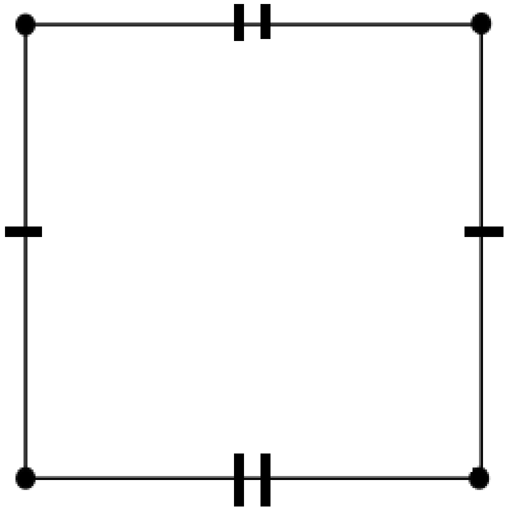

Torus

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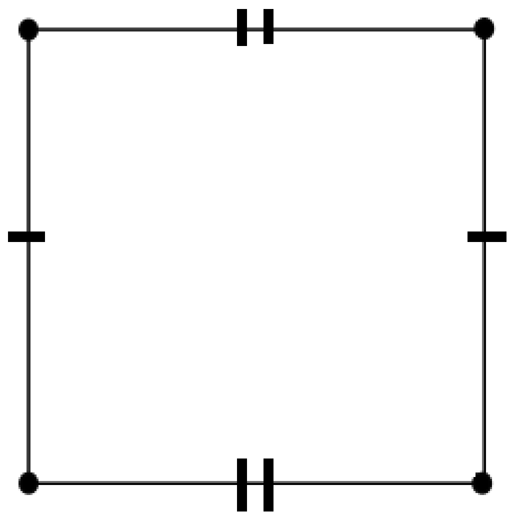

Torus

- Genus 1

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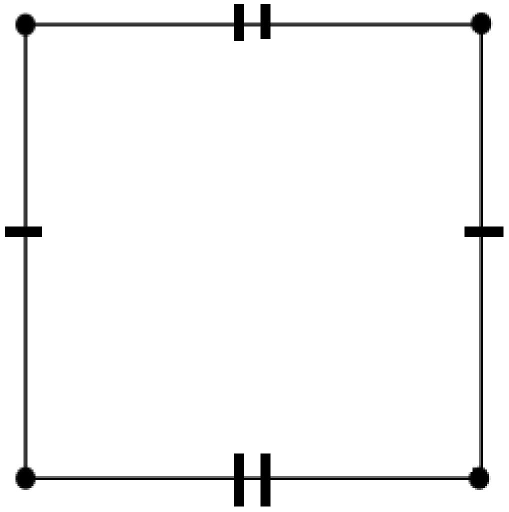
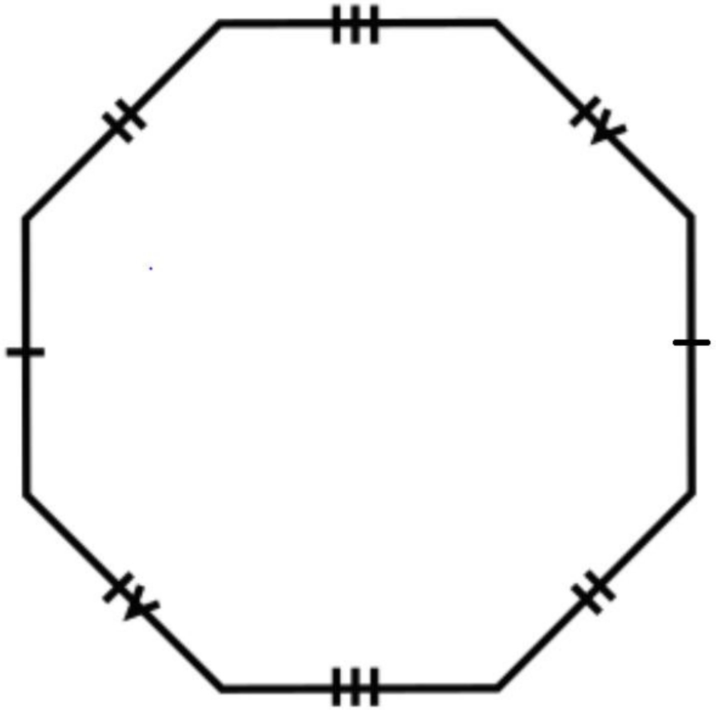


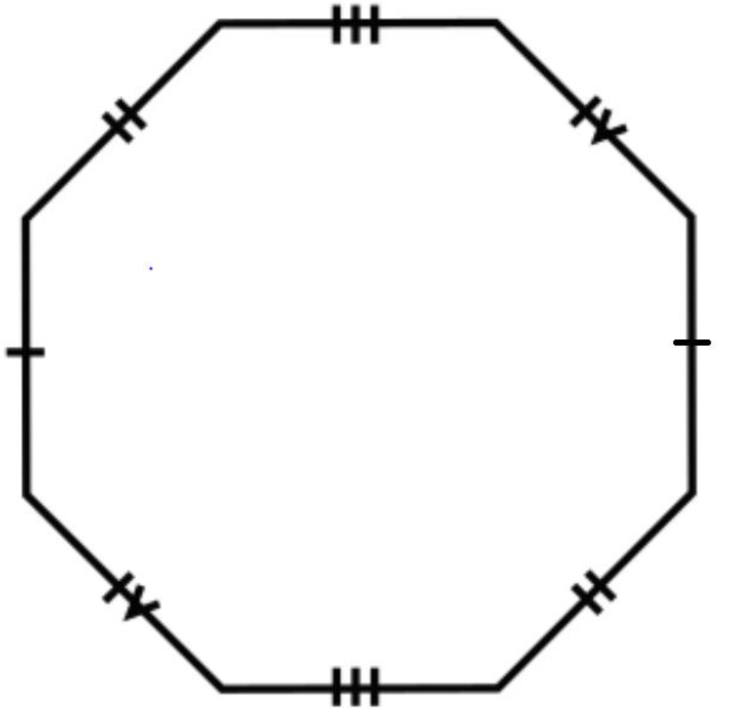
Torus
- Genus 1
- Flat geometry everywhere.

Octagon



Regular Octagon:

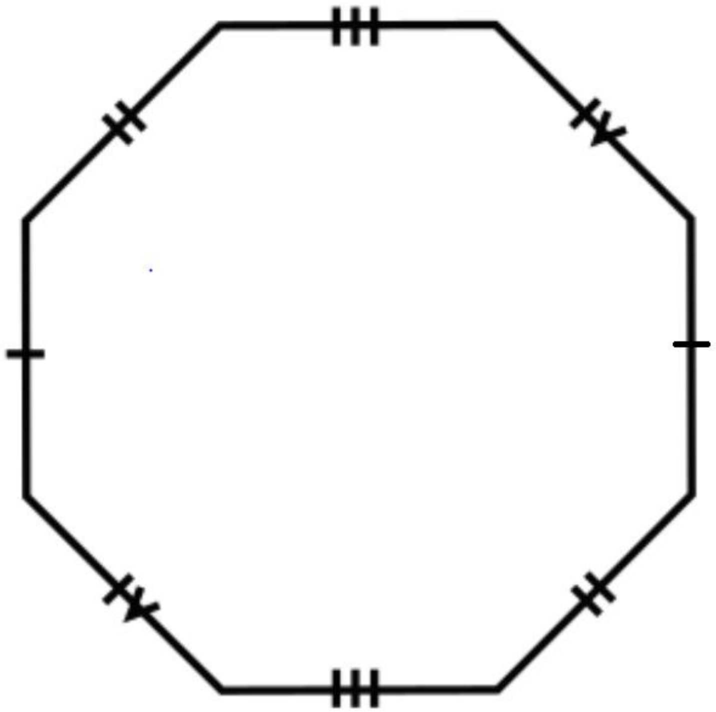
Octagon



Regular Octagon:

- Genus 2

Octagon

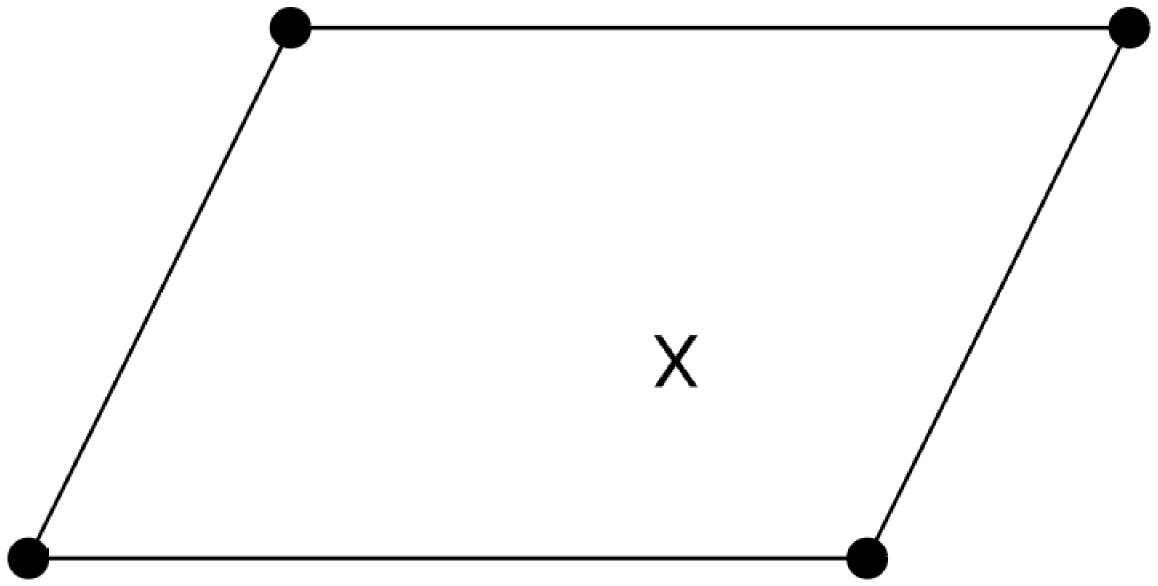


Regular Octagon:

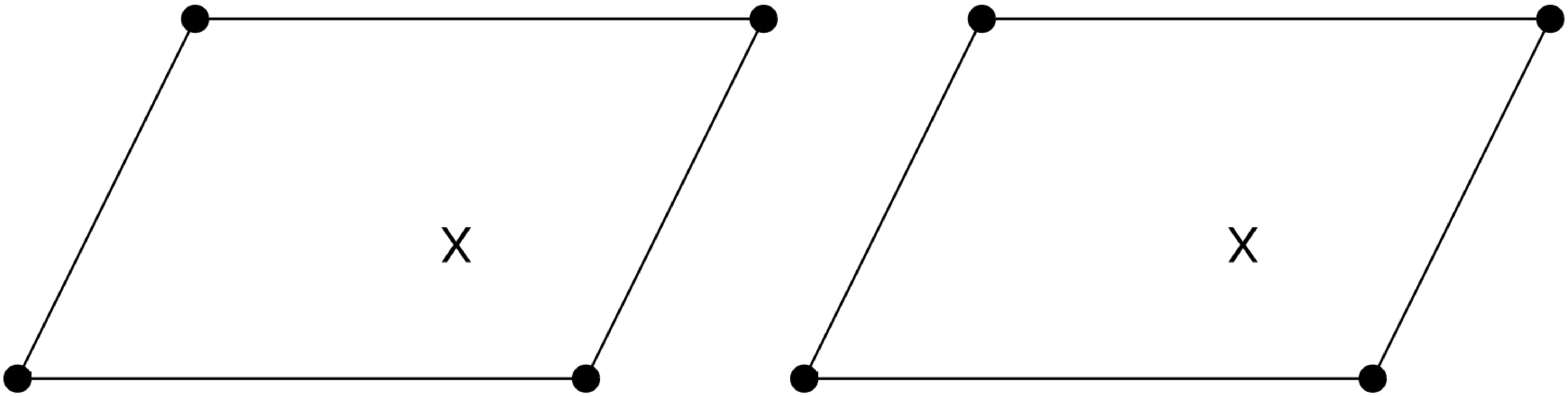
- Genus 2
- Single cone point of angle 6π

Doubled slit torus construction

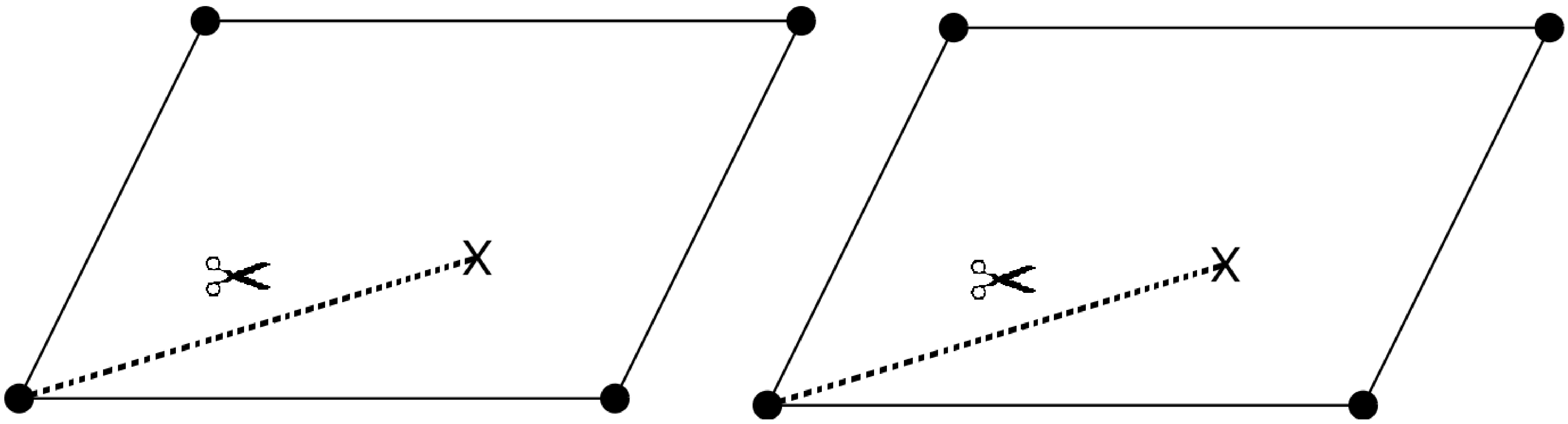
Take a flat torus and mark two points



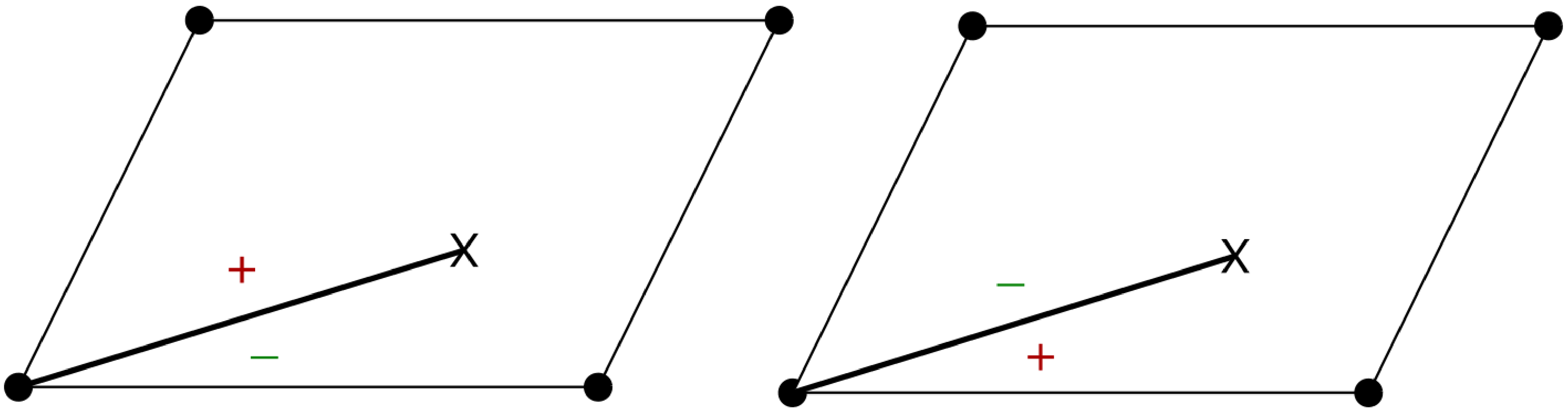
Take an identical copy of the twice-marked torus



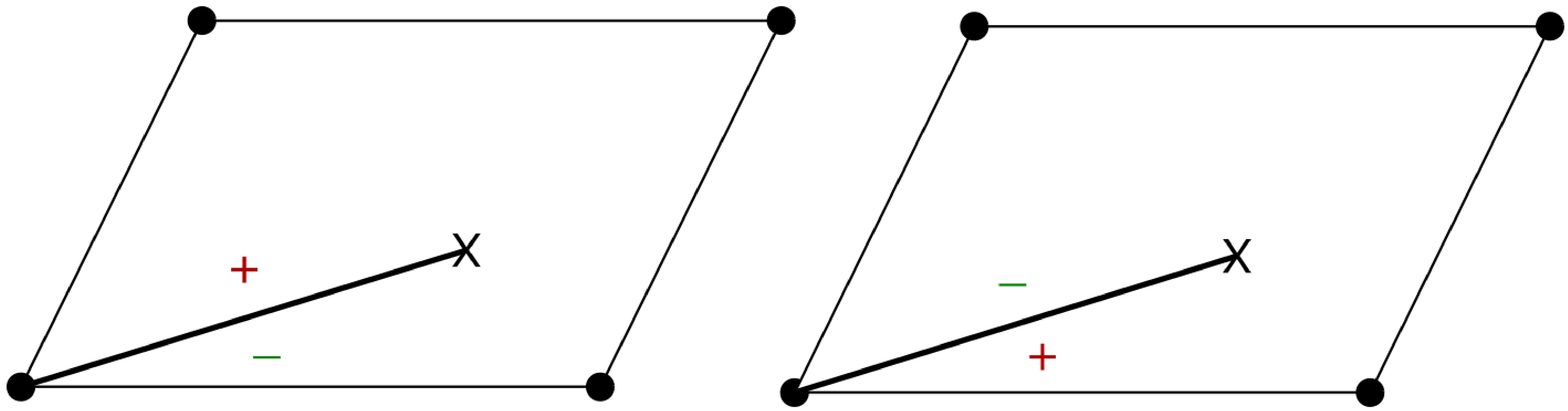
Cut a slit between the marked points



Glue opposite sides of the slit together



Doubled Slit Torus



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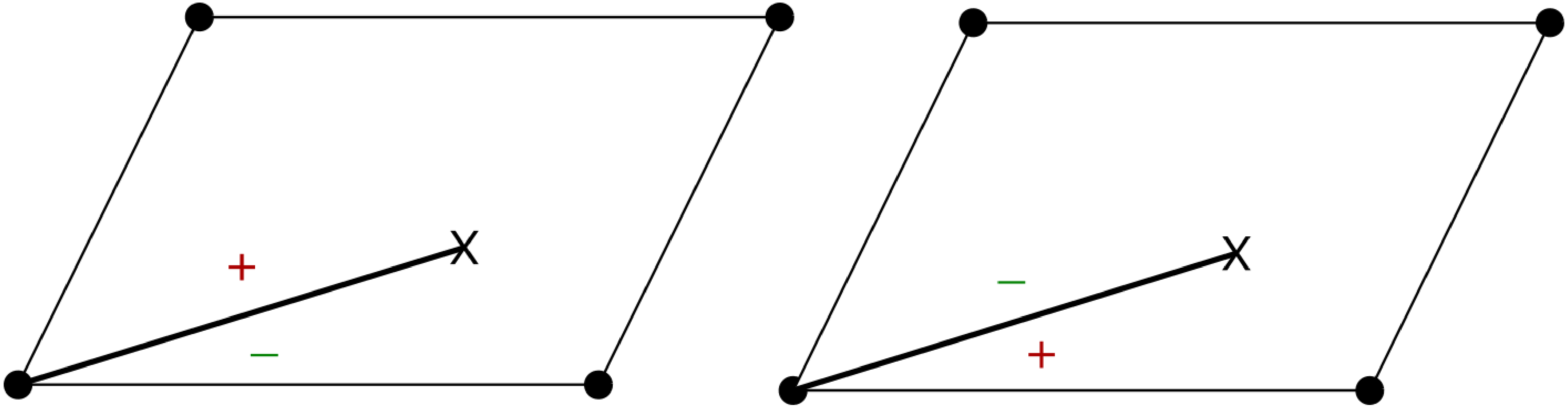
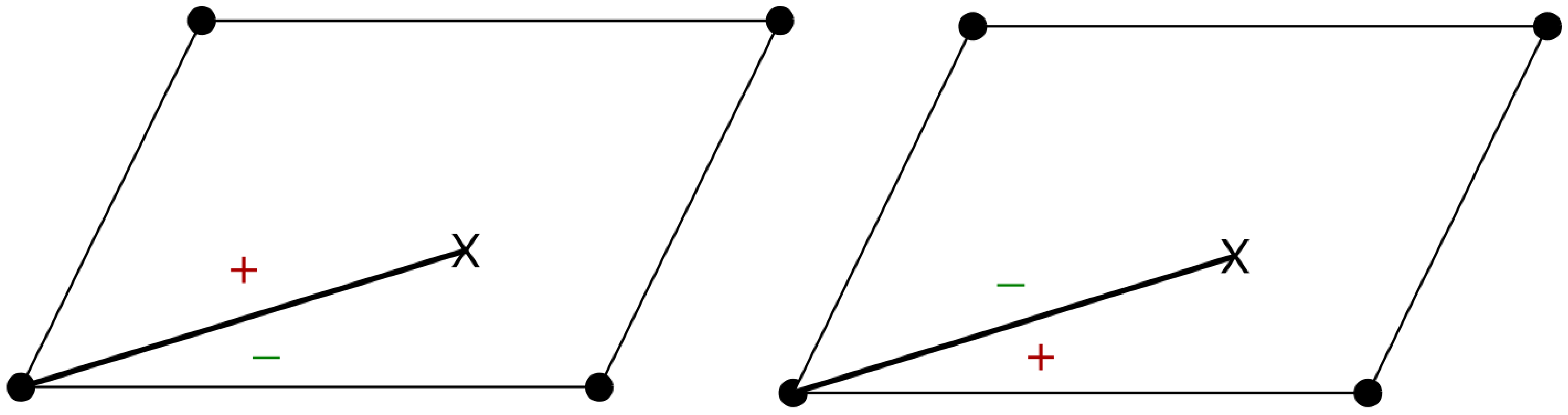


Genus 2 surface

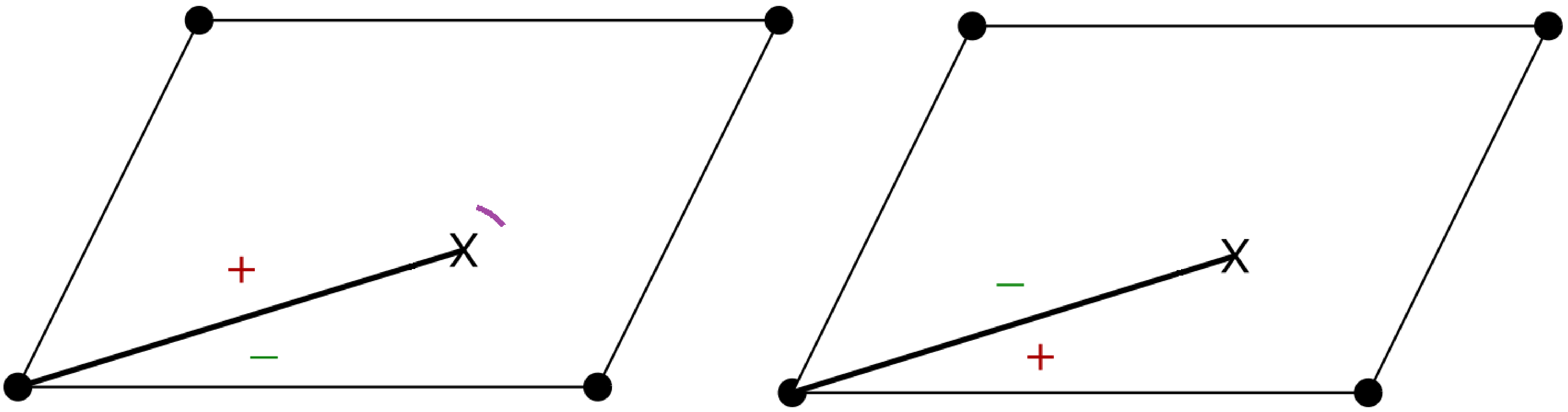
Doubled Slit Torus



Genus 2 surface

2 cone type singularities of angle 4π

Doubled Slit Torus



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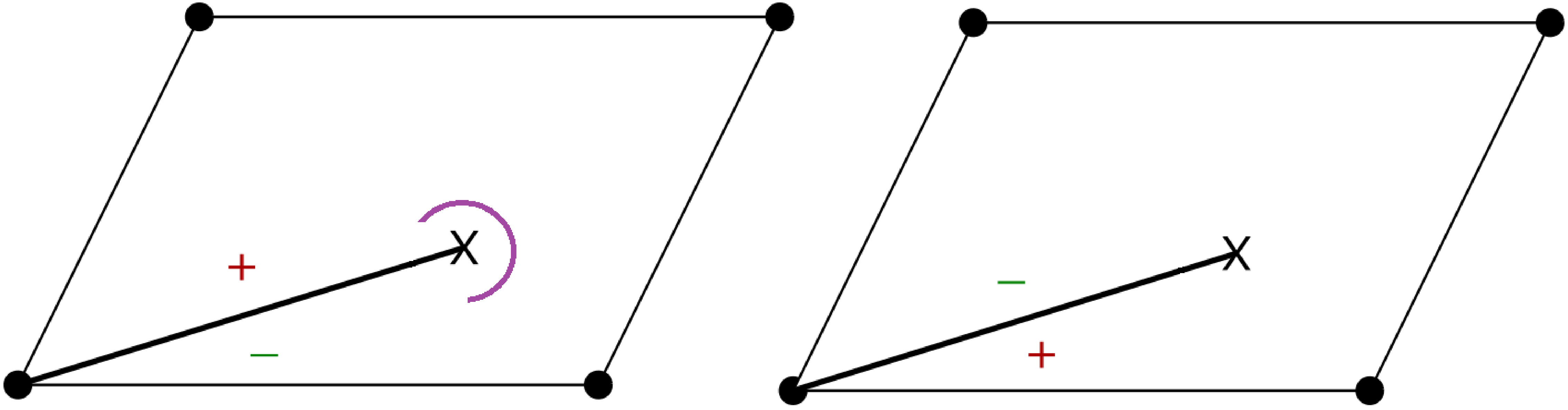

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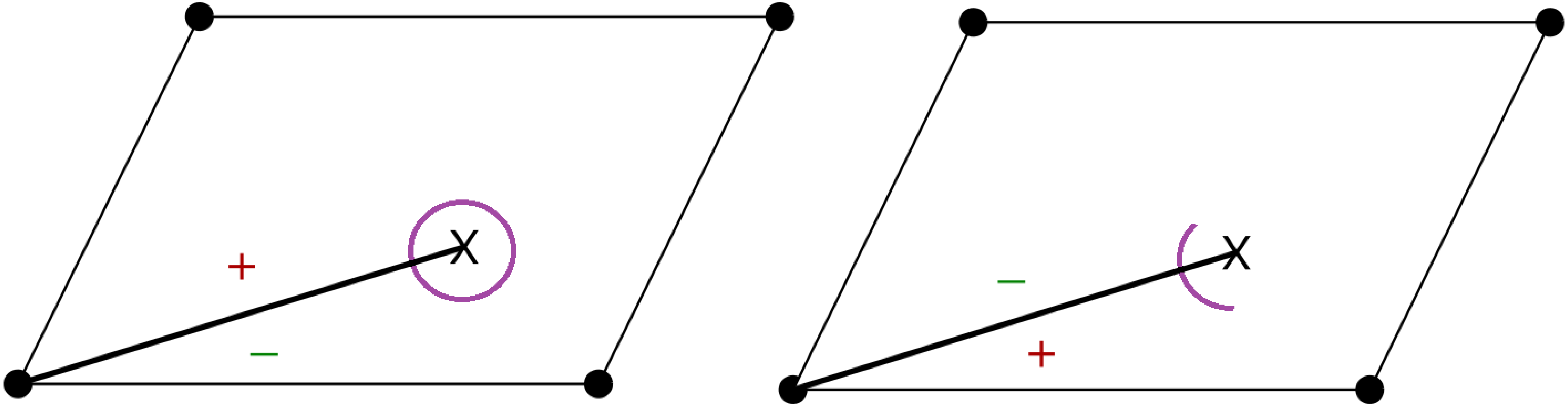

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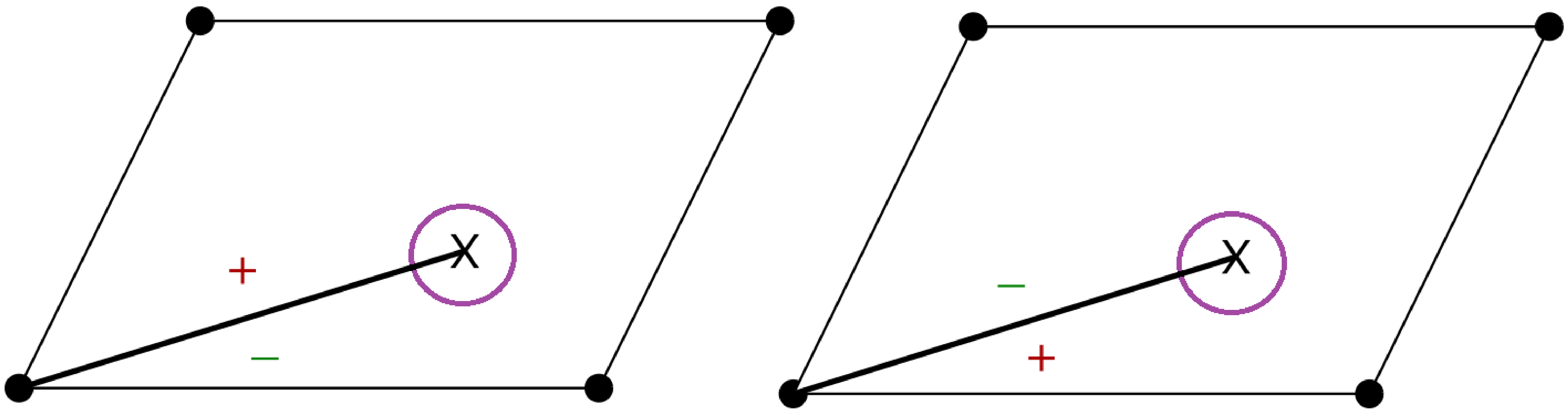

Genus 2 surface

2 cone type singularities of angle 4π

Why doubled slit tori?

(Topology)

Are a natural construction of a higher genus surface from genus 1 surfaces.

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(Dynamics)

First higher genus surface with minimal but not uniquely ergodic straight-line flow.

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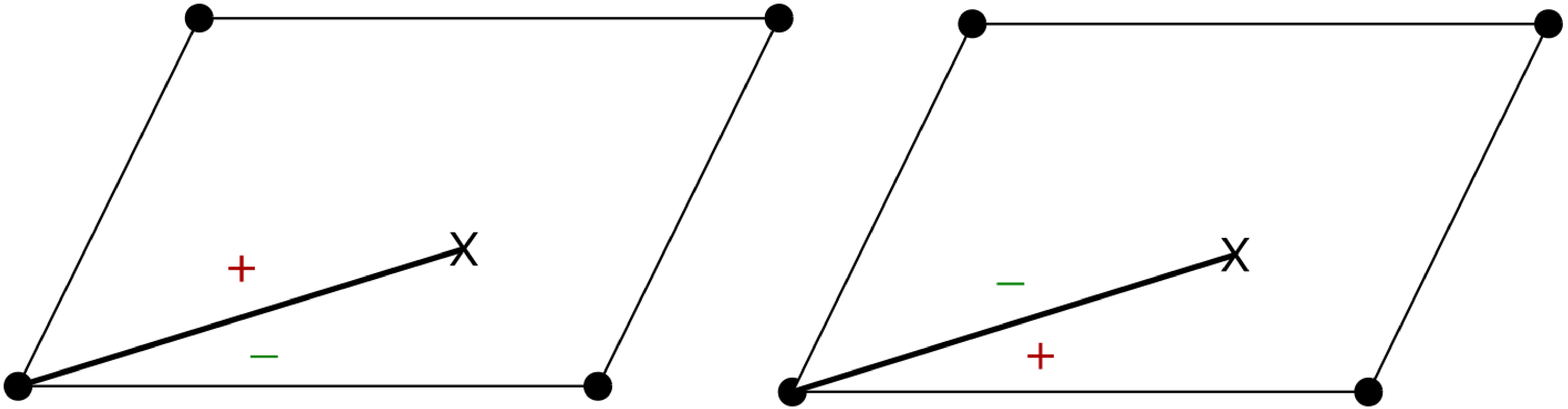
(Geometry)

Are examples of translation surfaces.

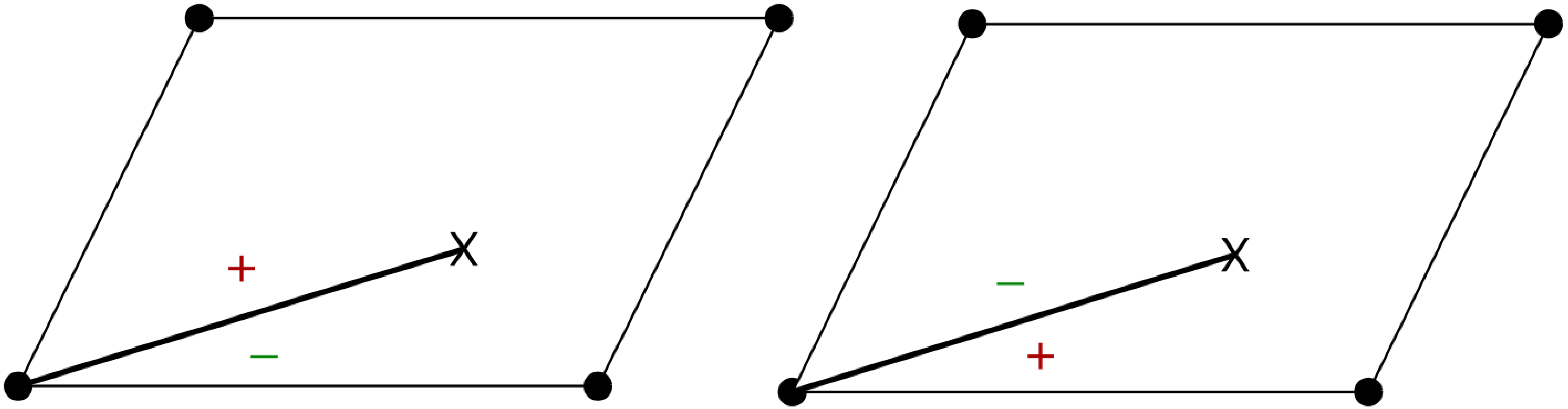
Translation structure

Embedding into complex plane endows the surface with a Riemann surface structure X



Translation structure

Embedding into complex plane endows the surface with a Riemann surface structure X and the holomorphic differential dz.



Translation surfaces

More generally any pair (X, ω) where X is a Riemann surface and ω is a non-zero holomorphic differential is called a ***translation surface***.

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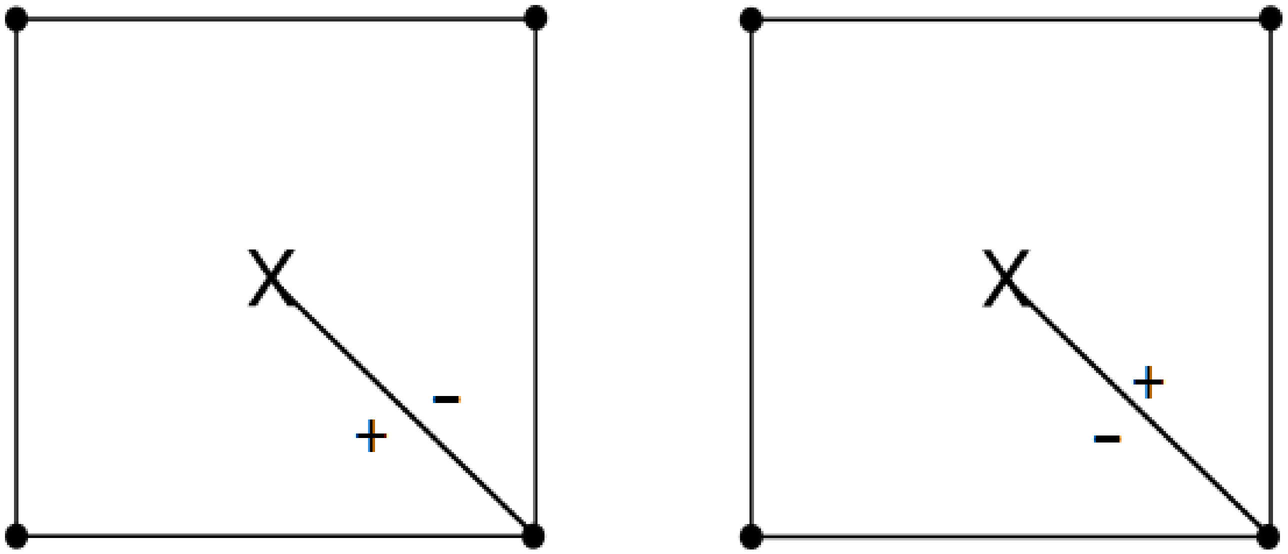
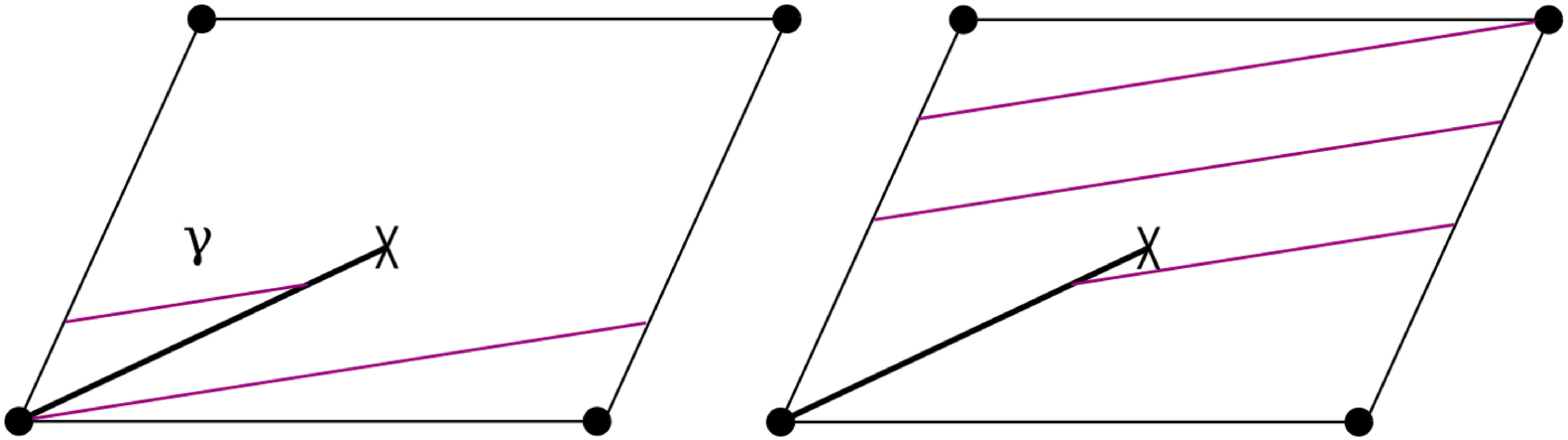
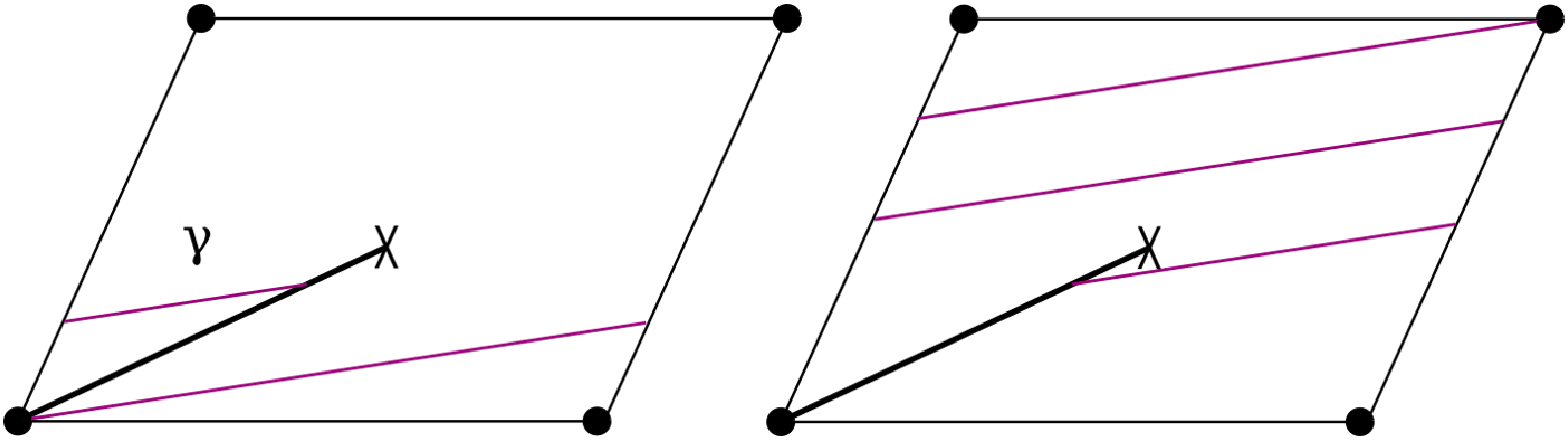
The holomorphic differential allows us to measure lengths and gives a sense of direction.

We are interested in paths on doubled slit tori

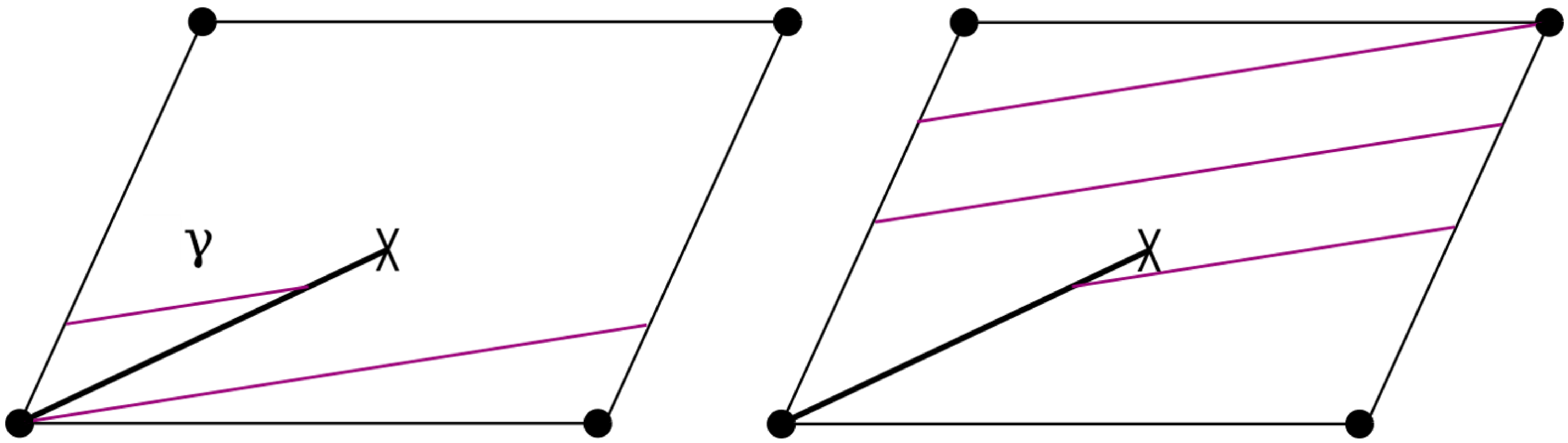


A ***saddle connection*** is a straight-line trajectory starting and ending at a cone type singularity.





Associated to each saddle connection is the ***holonomy vector***.



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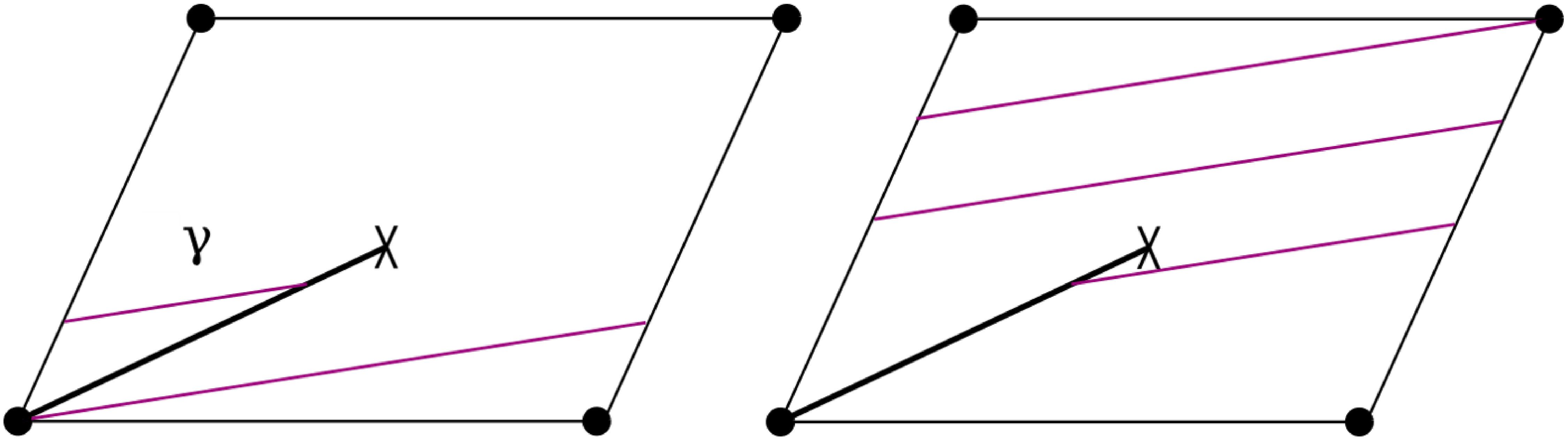
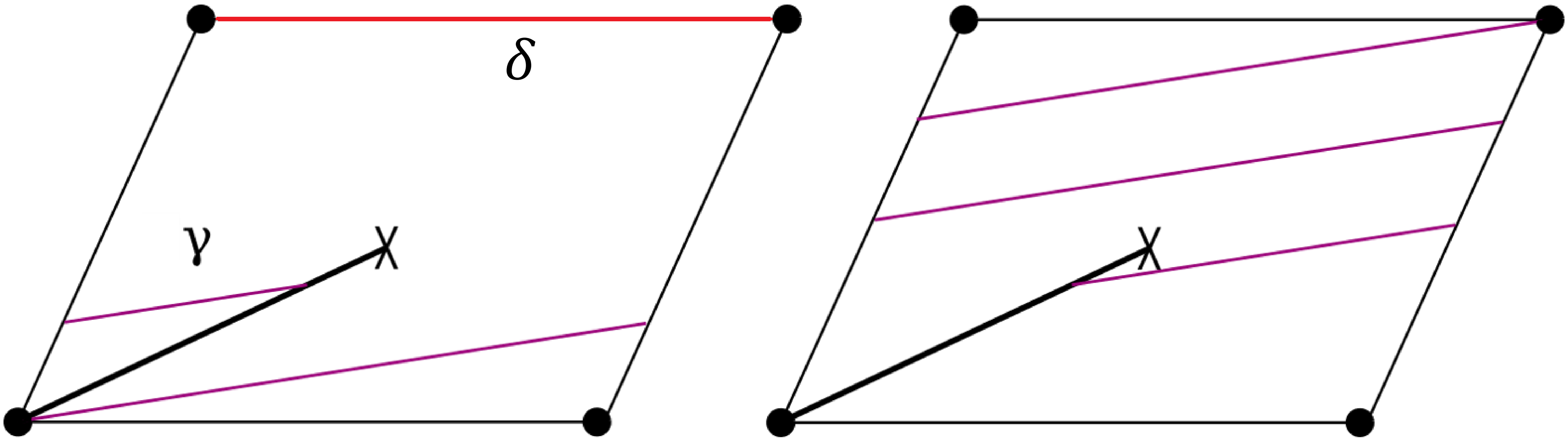
$$\int_\gamma dz = 4 + i$$



Associated to each saddle connection is the ***holonomy vector***.

$$\int_\gamma \, dz = 4 + i \text{ or } \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$


δ
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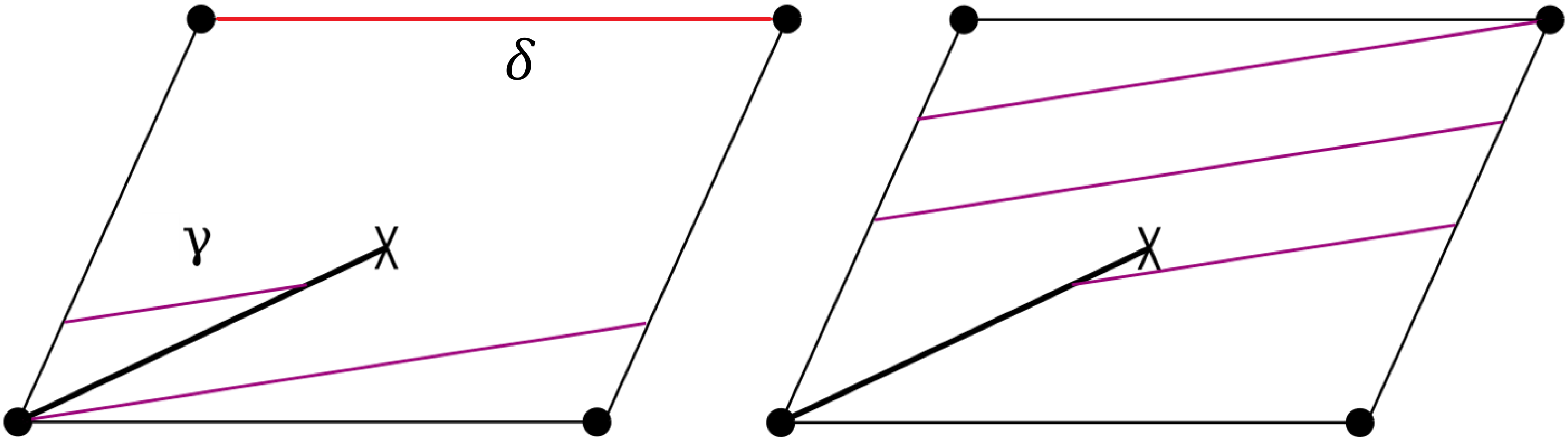
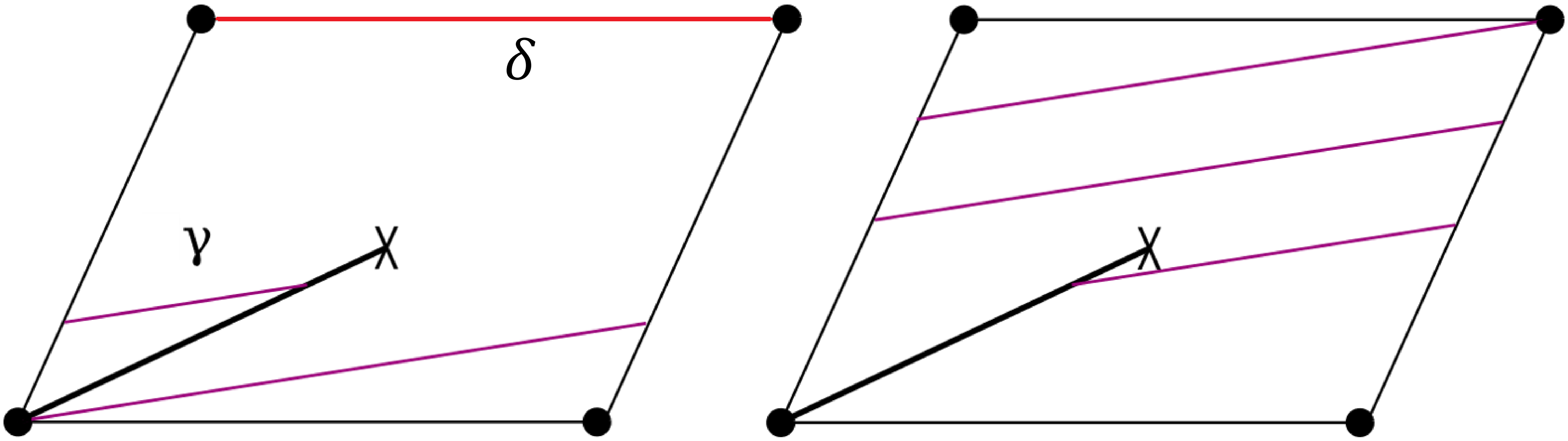


$$\int_{\delta} dz = 1 + 0i \text{ or } \binom{1}{0}$$



$$\int_\gamma dz = 4 + i \text{ or } \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \int_\delta dz = 1 + 0i \text{ or } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let Λ_ω denote the set of all holonomy vectors.

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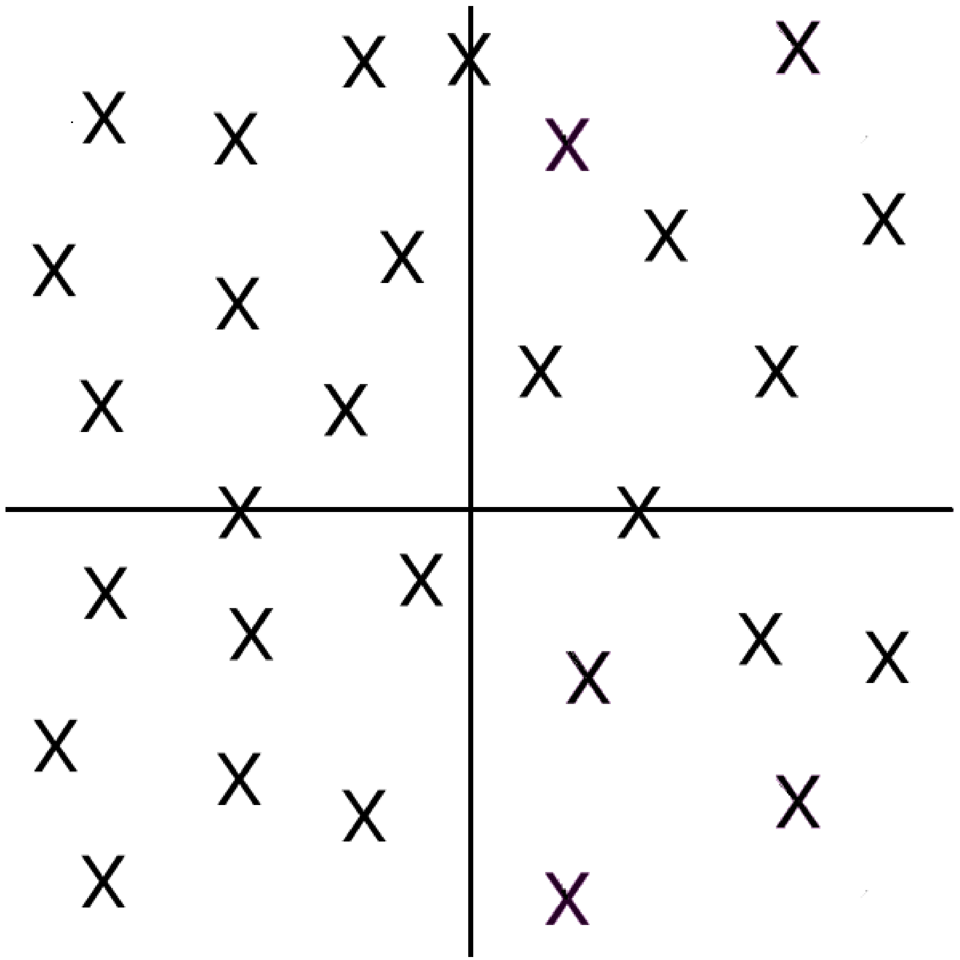
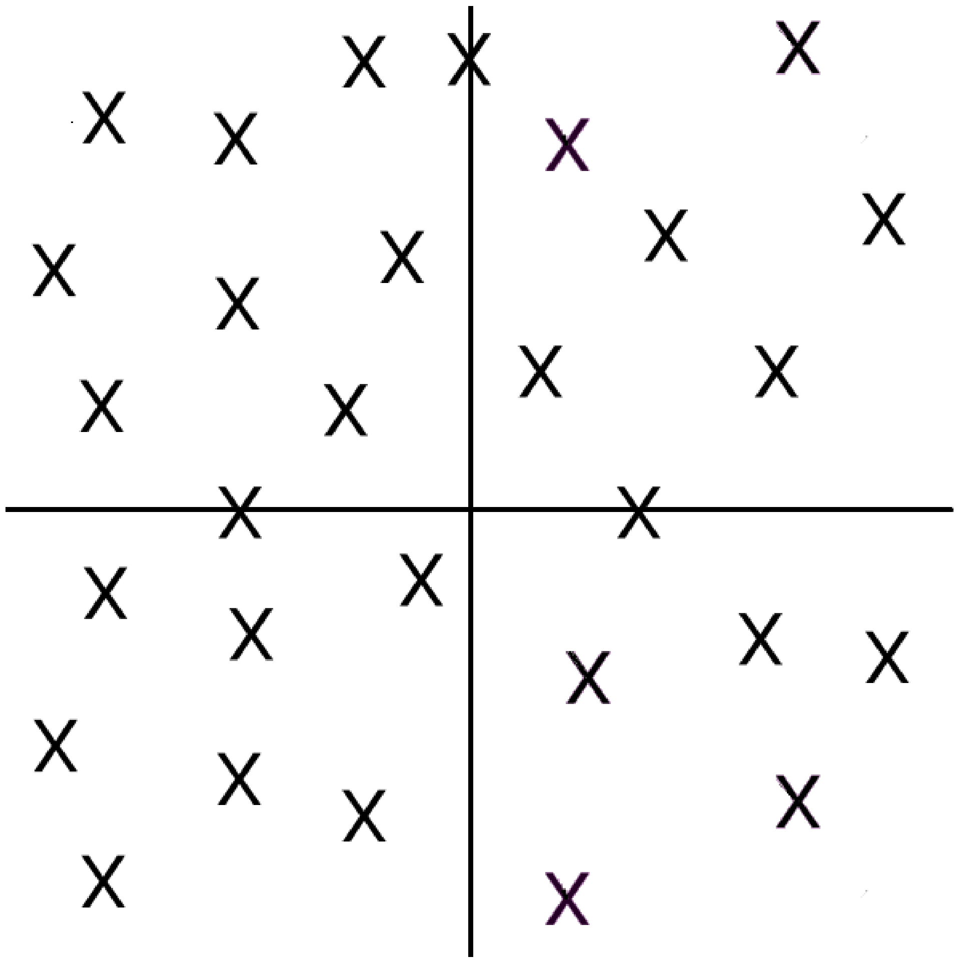


Discreteness

Λ_ω

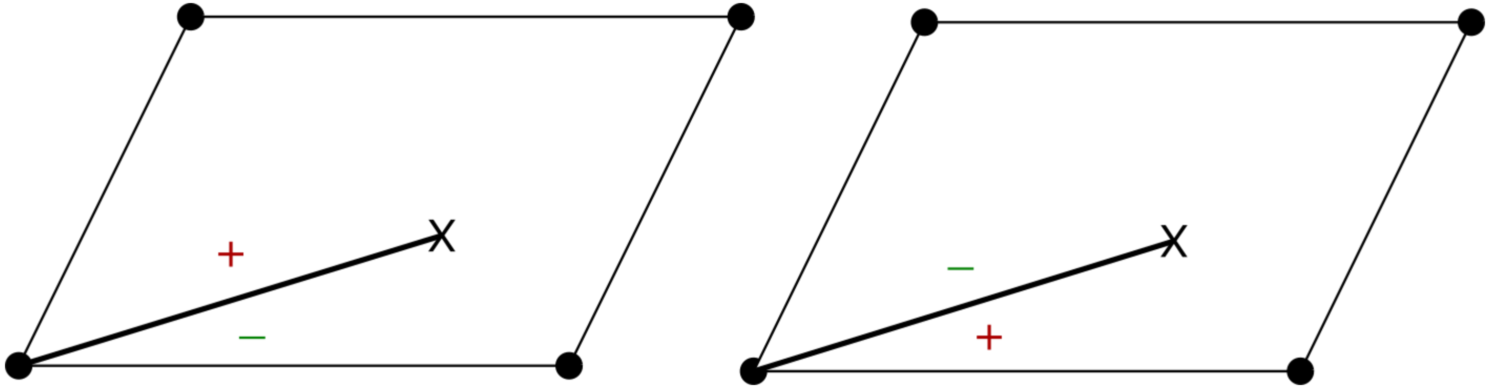


Let Λ_ω denote the set of all holonomy vectors.

Veech: Λ_ω is a discrete subset!

How random are the holonomy vectors?

(X, ω)



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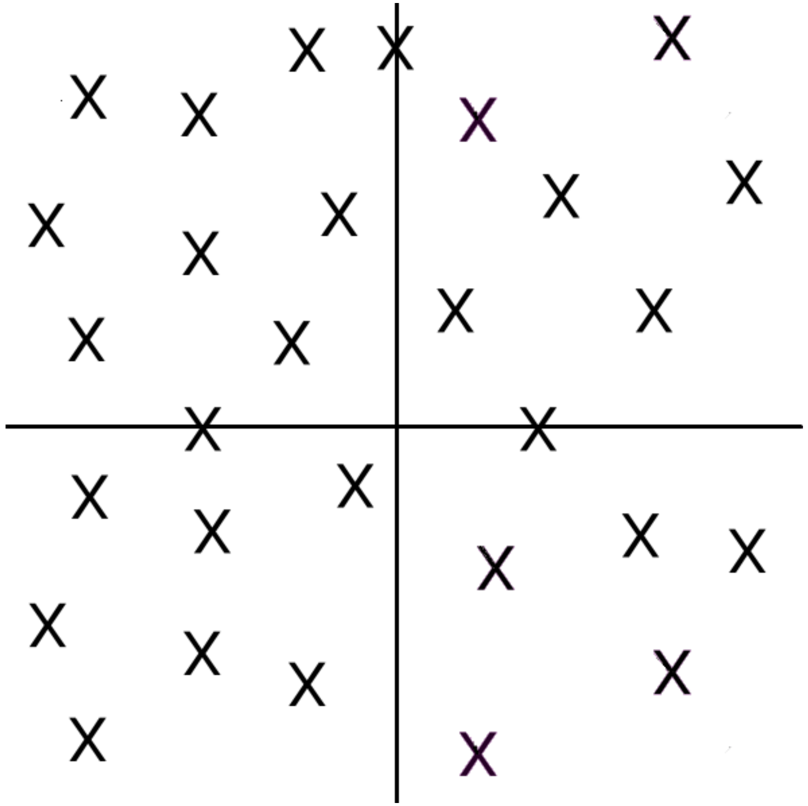

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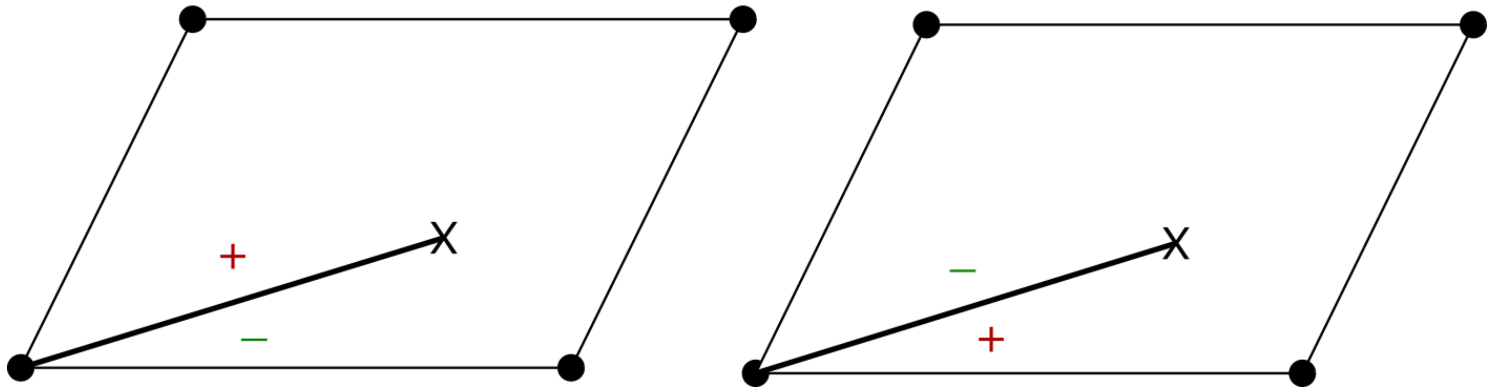

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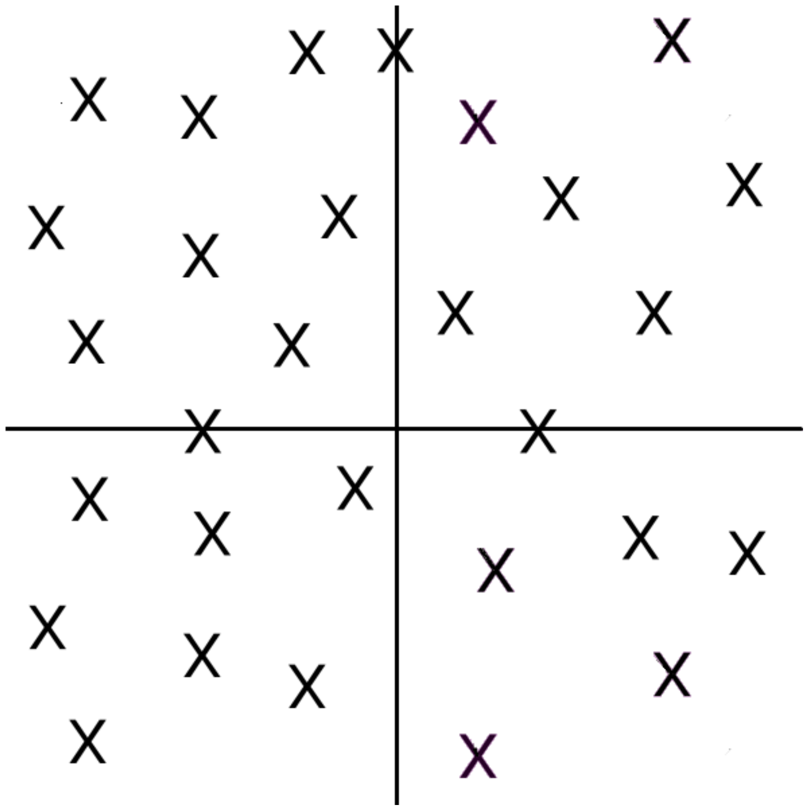
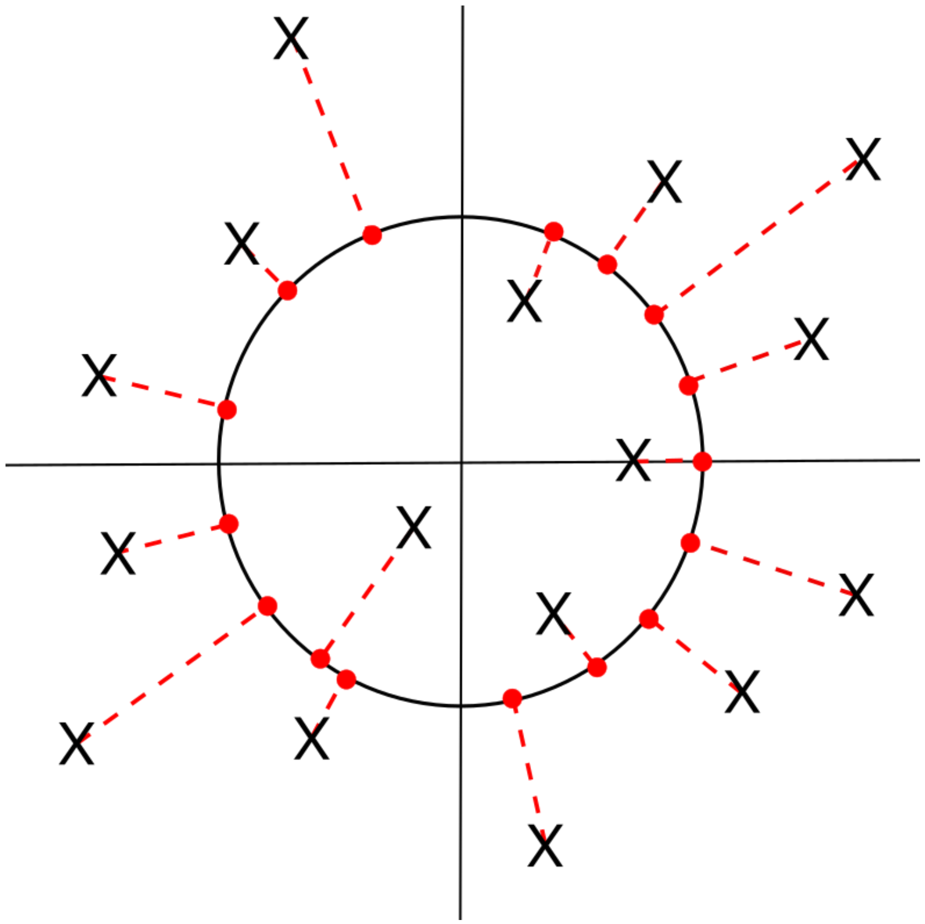


Angles as a test of randomness



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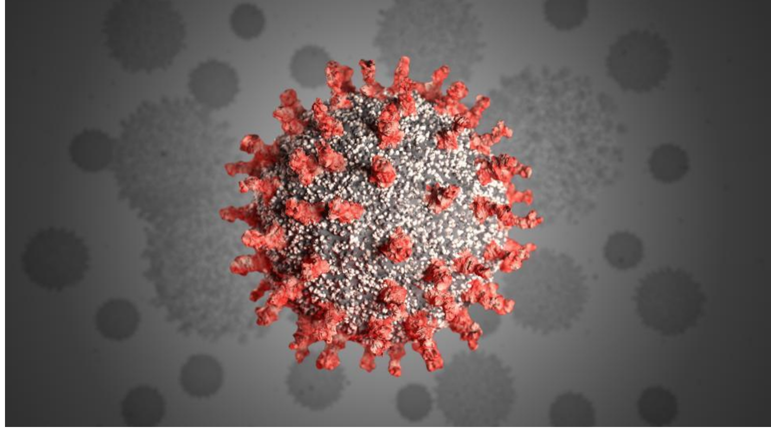

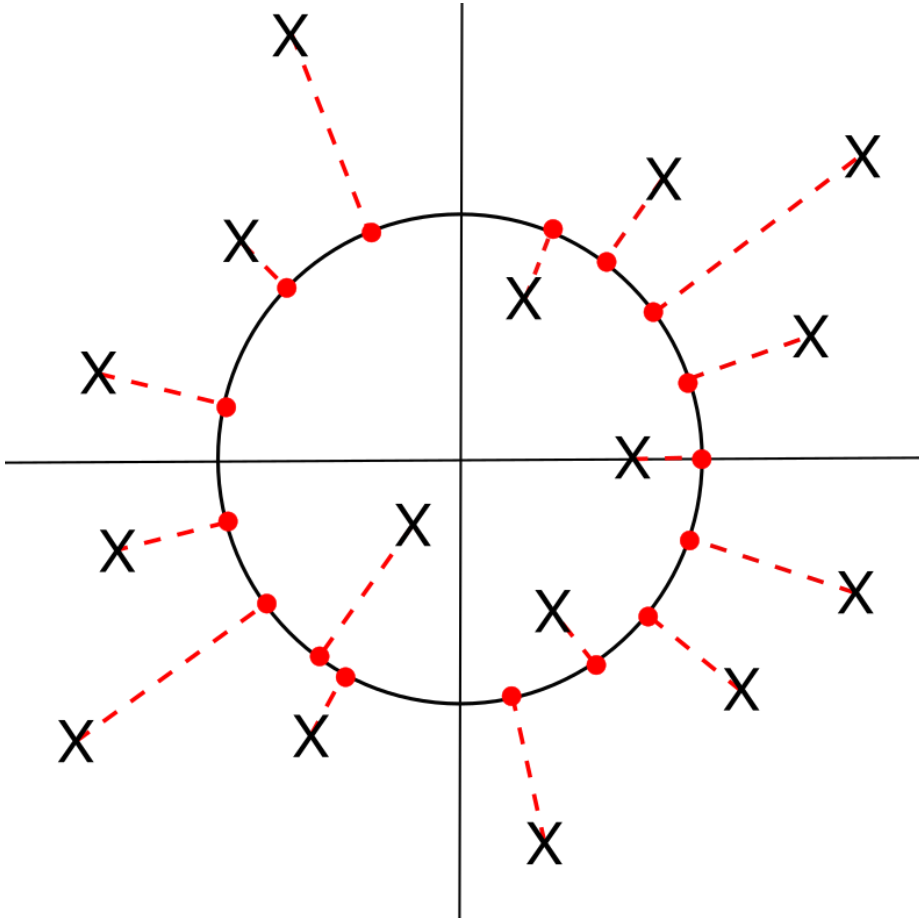

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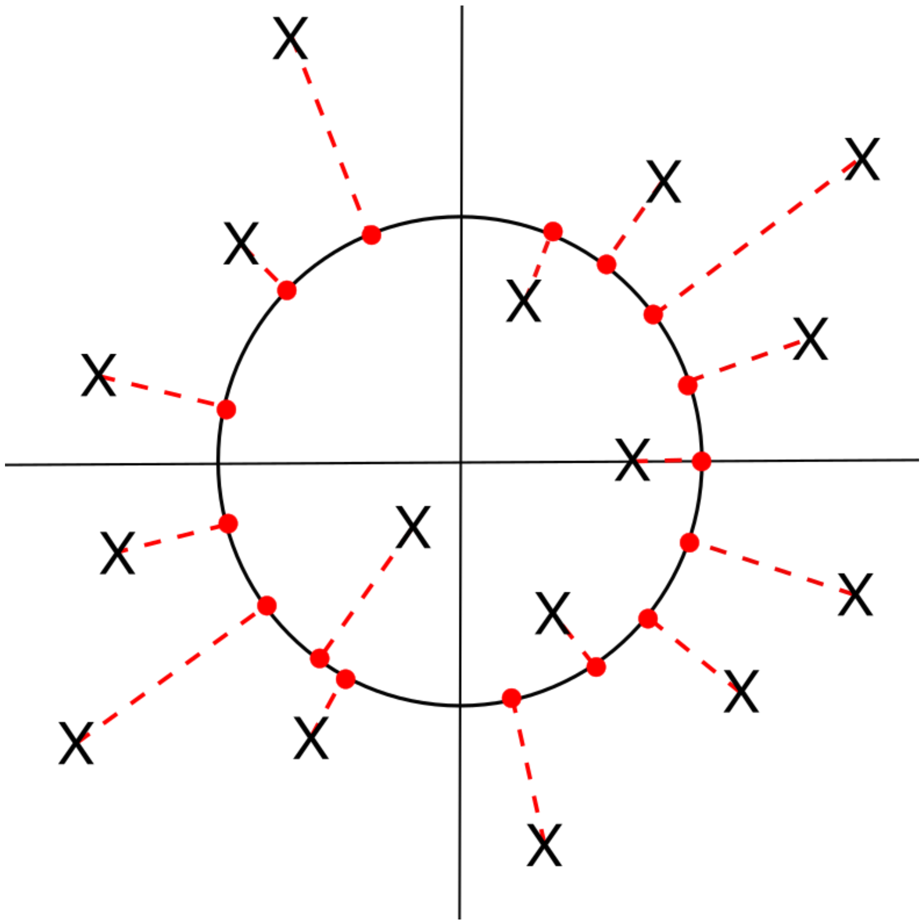
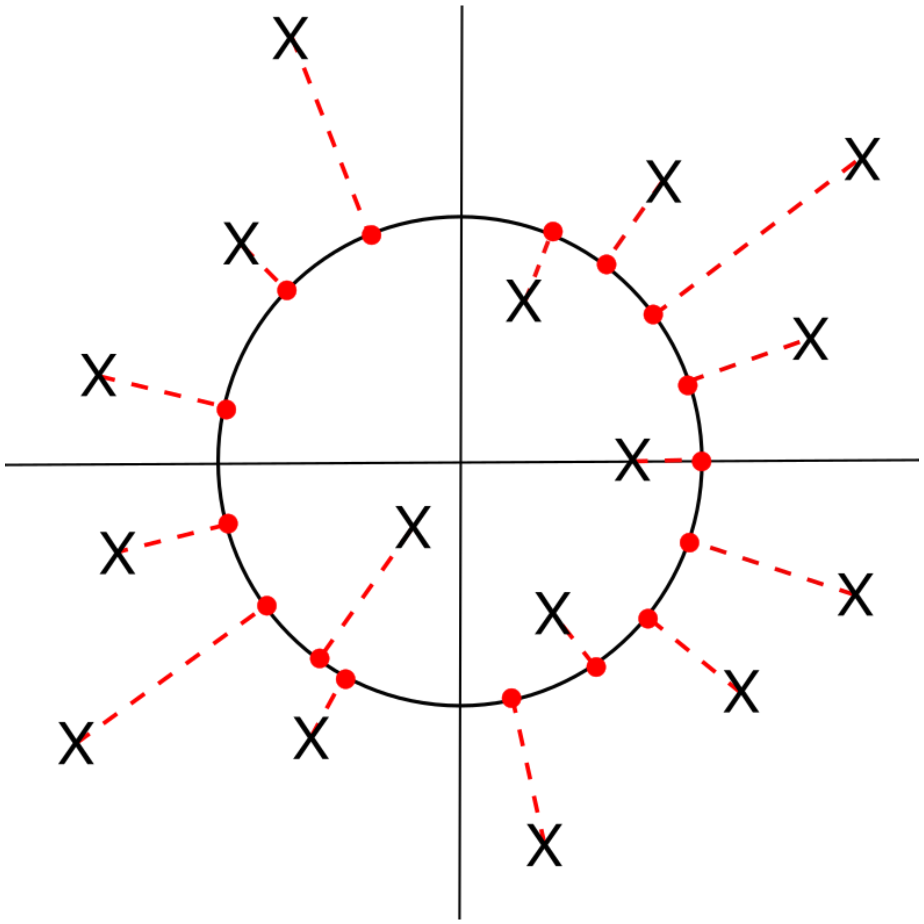


- *Masur*: angles are dense

Angles as a test of randomness



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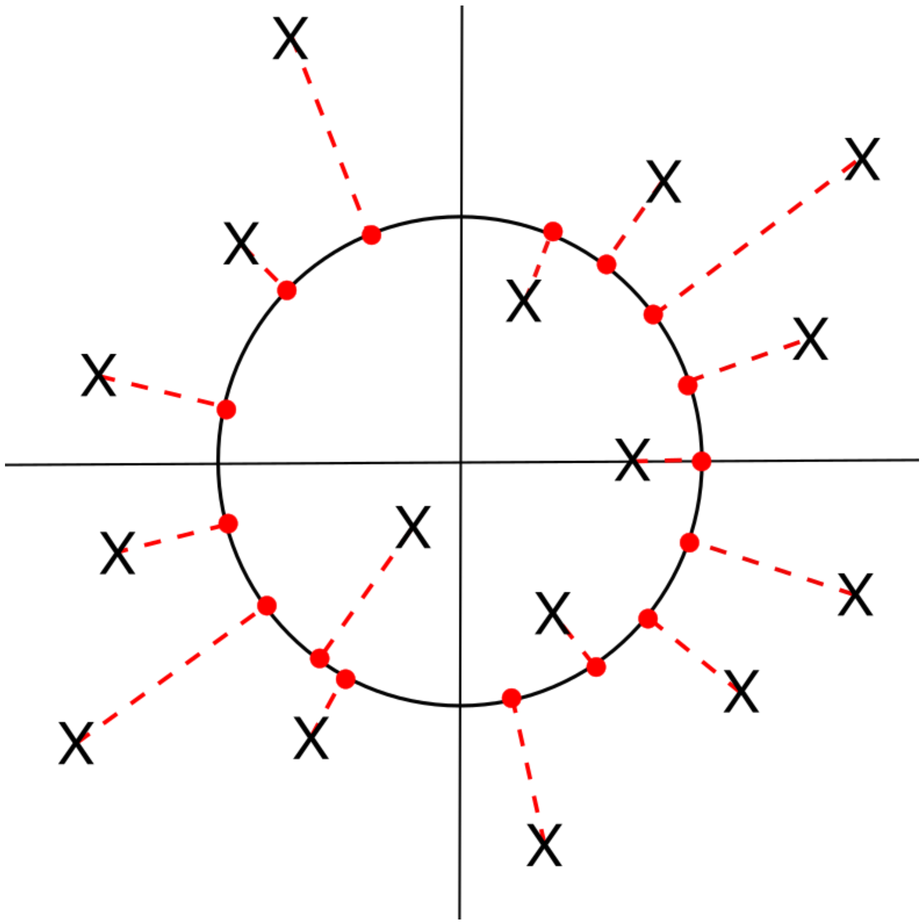
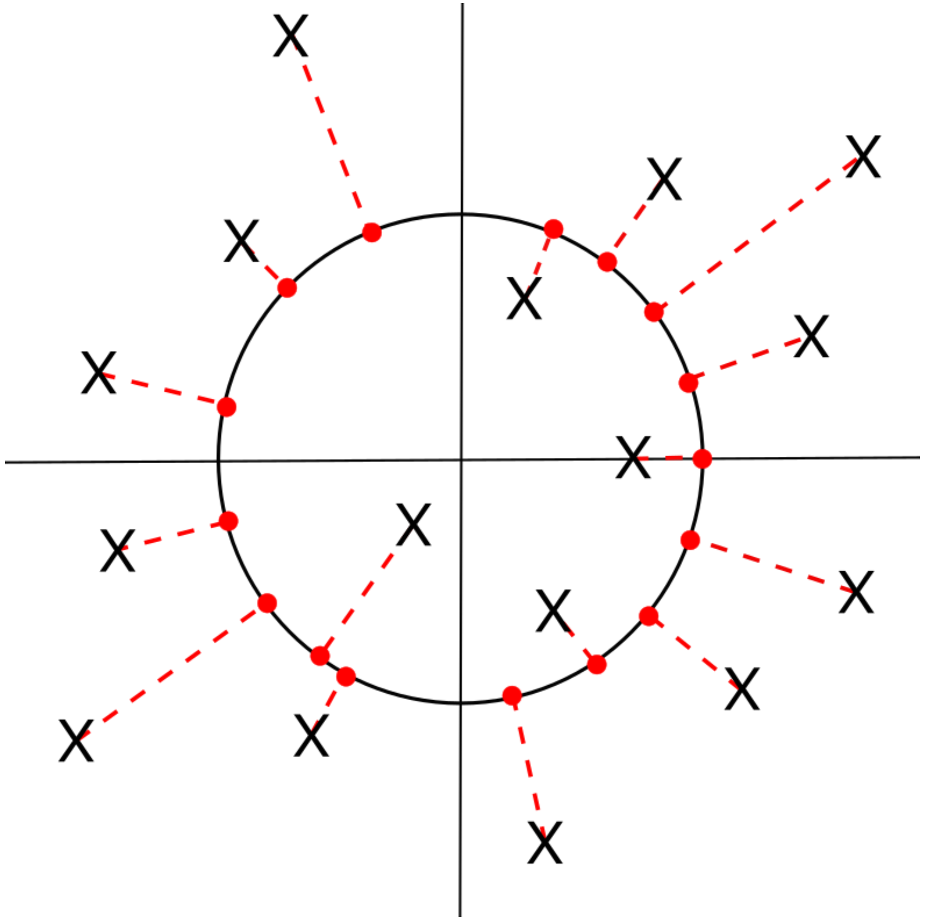


- *Masur*: angles are dense
- *Vorobets*: angles are ***equidistributed*** for almost every translation surface
- *Eskin-Marklof-Morris*: angles are ***equidistributed*** for covers of lattices surfaces



Upshot: Saddle connections appear to behave randomly at first glance.

A second test of randomness

A second test of randomness is to consider ***gaps*** of sequences.

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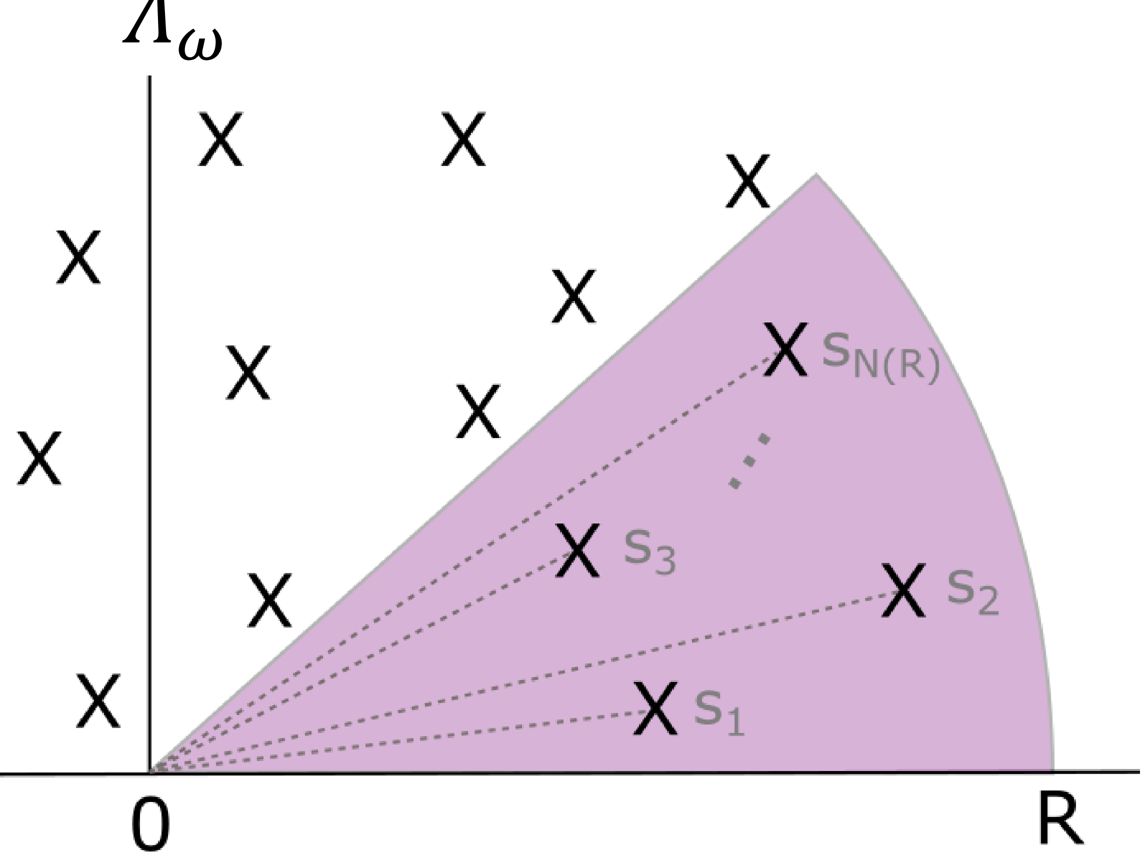
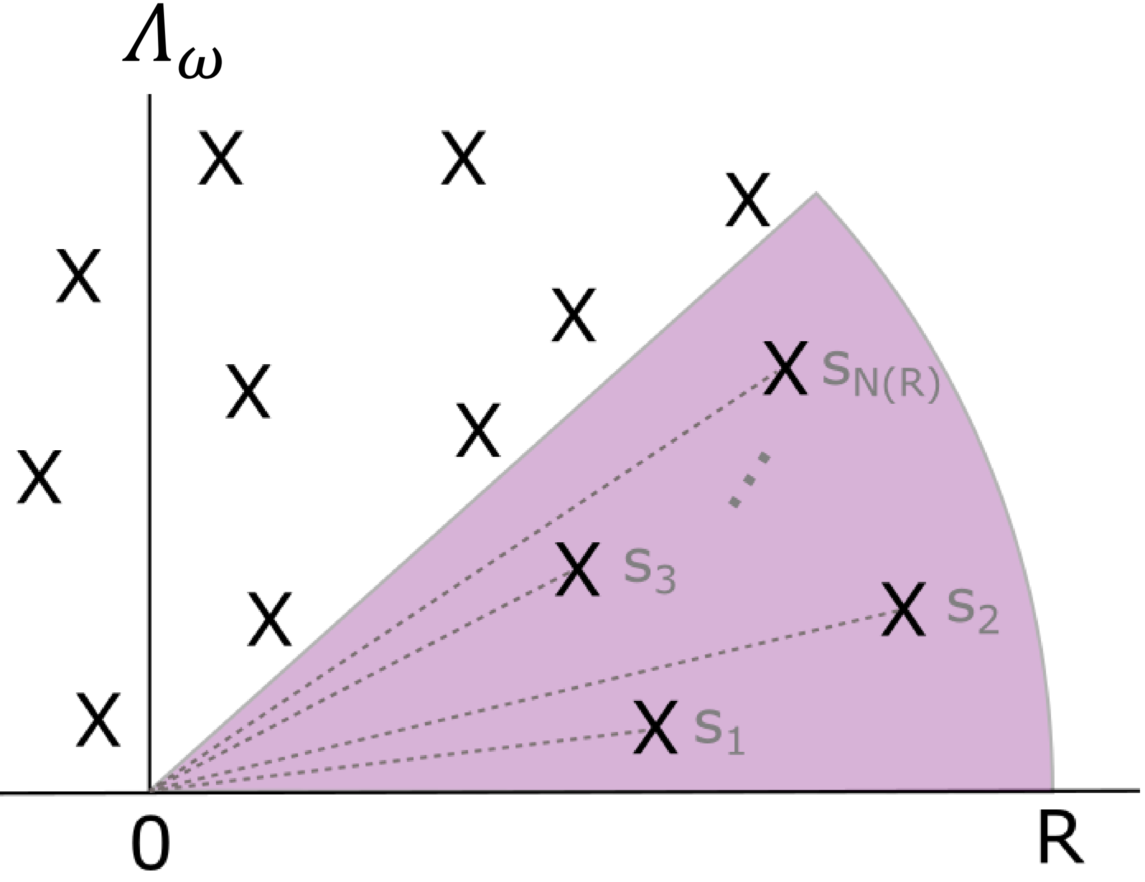
We consider slopes of saddle connections instead of angles.

Slopes of holonomy vectors



Let $Slopes^R(\Lambda_\omega)$ denote the slopes in an eighth sector up to length R.

Slopes of holonomy vectors

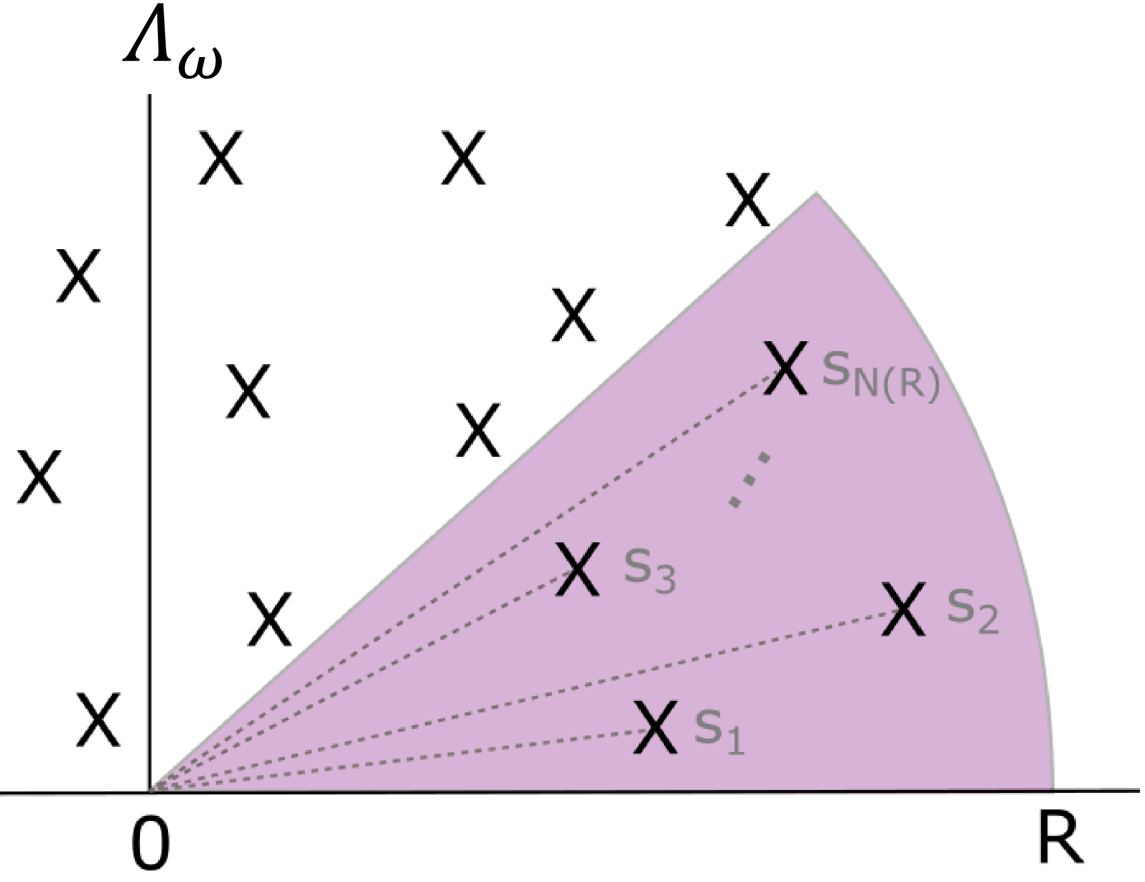


Let $Slopes^R(\Lambda_\omega)$ denote the slopes in an eighth sector up to length R.

$$Slopes^R(\Lambda_\omega) = \{s_0 = 0 < s_1 < \cdots < s_{N(R)}\}$$

where $N(R) = |Slopes^R(\Lambda_\omega)|$.

Slopes of holonomy vectors



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where $N(R) = |Slopes^R(\Lambda_\omega)|$.

Eskin-Masur showed $N(R) \sim R^2$.

Gaps of holonomy vectors

Consider the *gaps* of slopes

$$Gaps^R(\Lambda_\omega) = \{ \quad (s_i - s_{i-1}) \mid i = 1, \ldots, N(R) \}$$

Gaps of holonomy vectors

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Gaps of holonomy vectors

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$$Gaps^R(\Lambda_\omega) = \{R^2(s_i - s_{i-1}) |\ i = 1, \dots, N(R)\ \}$$

What can we say about the distribution of gaps?

Gap distribution

The *gap distribution* is given by

$$Gaps^R(\Lambda_\omega)$$

Gap distribution

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$$Gaps^{R}(\Lambda_{\omega}) \cap I$$

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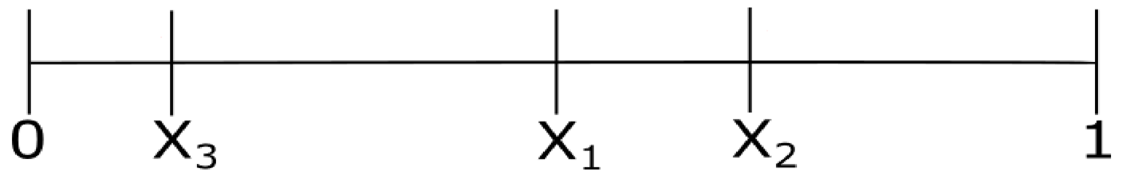
This measures the proportion of gaps in an interval I.

What can we say about this limit? What do we expect?

Context from probability

Suppose that $(X_i)_{i=1}^{\infty}$ are a sequence of IID random variables uniformly distributed on [0,1].



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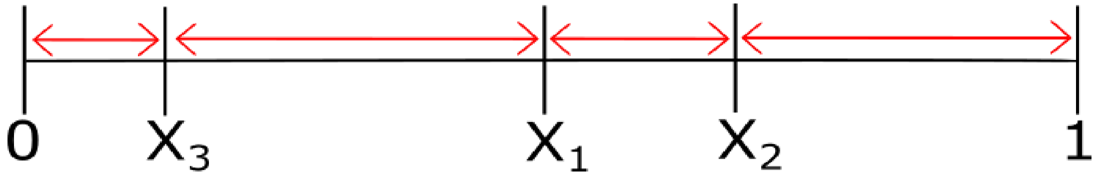

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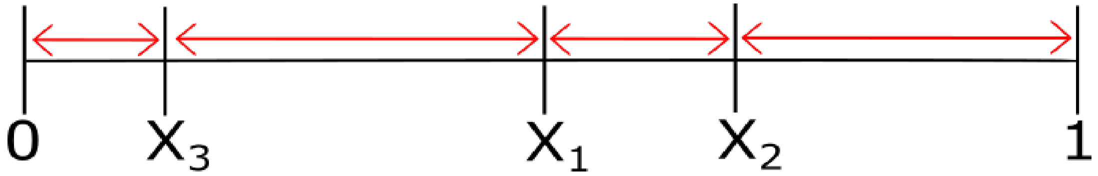

$$Gaps\{(X_i)_{i=1}^{n}\}$$

Context from probability

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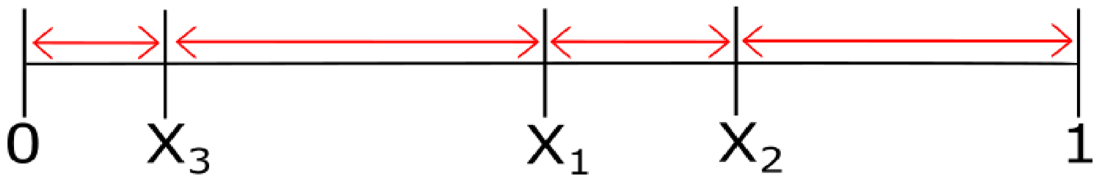
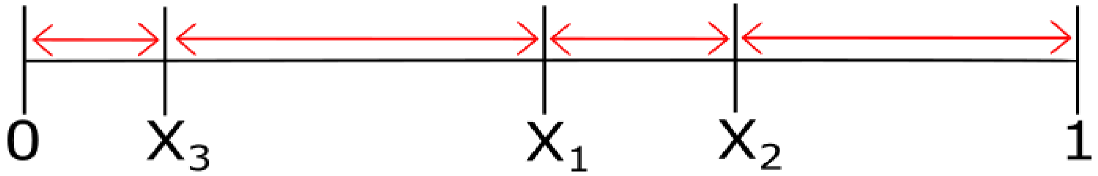


$$\frac{|Gaps\{(X_i)_{i=1}^{n}\} \cap I|}{n}$$

Context from probability

Suppose that $(X_i)_{i=1}^{\infty}$ are a sequence of IID random variables uniformly distributed on [0,1].

The associated gaps are **exponential**.

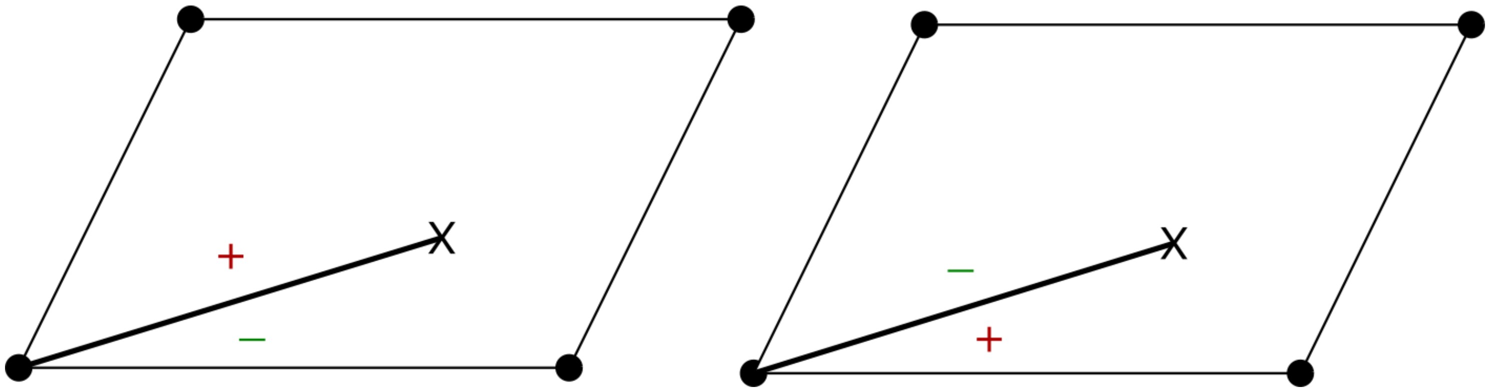


$$\frac{\left|Gaps\{(X_i)_{i=1}^{n}\}\cap I\right|}{n} \to \int_{\mathrm{I}} e^{-x}\, dx$$

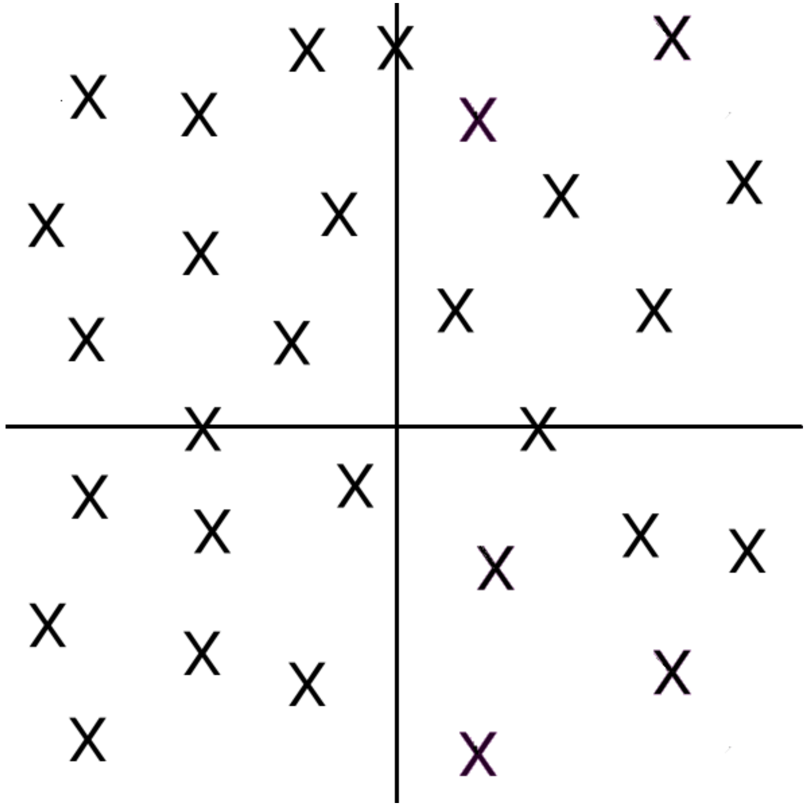
Theorem (S. 2020)

The gap distribution of almost every doubled slit torus is **not** exponential.

(X, ω)



Λ_ω

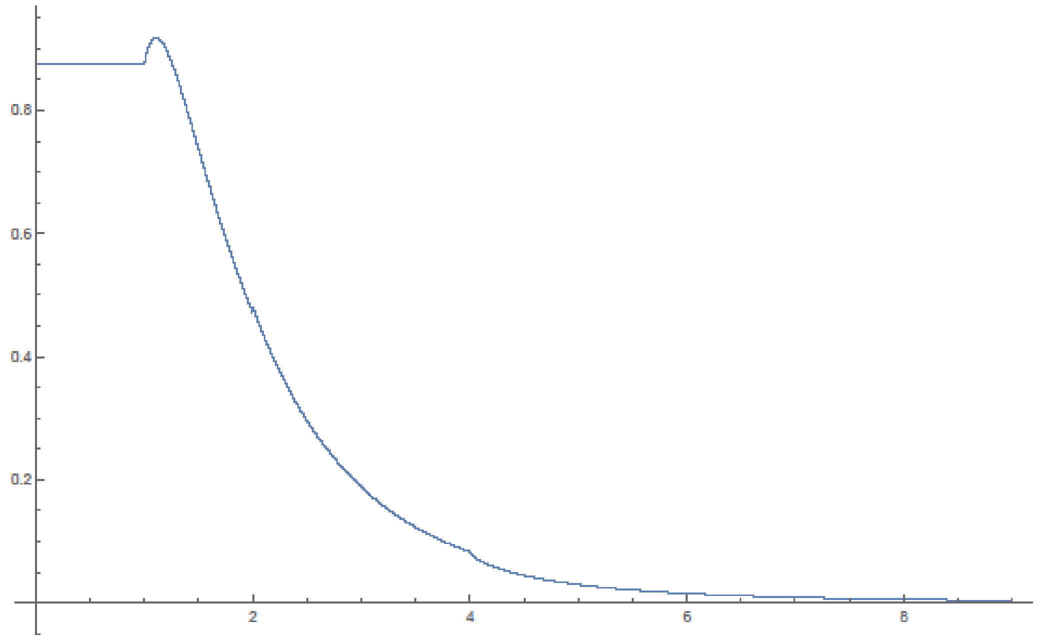


Theorem (S. 2020)

There exists a density function f so that

$$\lim_{R\to\infty} \frac{|Gaps^R(\Lambda_\omega) \cap I|}{N(R)} = \int_I f(x)\, dx$$

for almost every doubled slit torus.



Large gaps

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$$\int_t^{\infty} f(x)\, dx \sim t^{-2}.$$

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Compare with the IID case:

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Thus, large gaps are unlikely, but still much more likely than the random case!

Small gaps

The gap distribution has *support at zero:*

$$\int_0^{\varepsilon} f(x)\,dx > 0$$

for every $\varepsilon > 0$.

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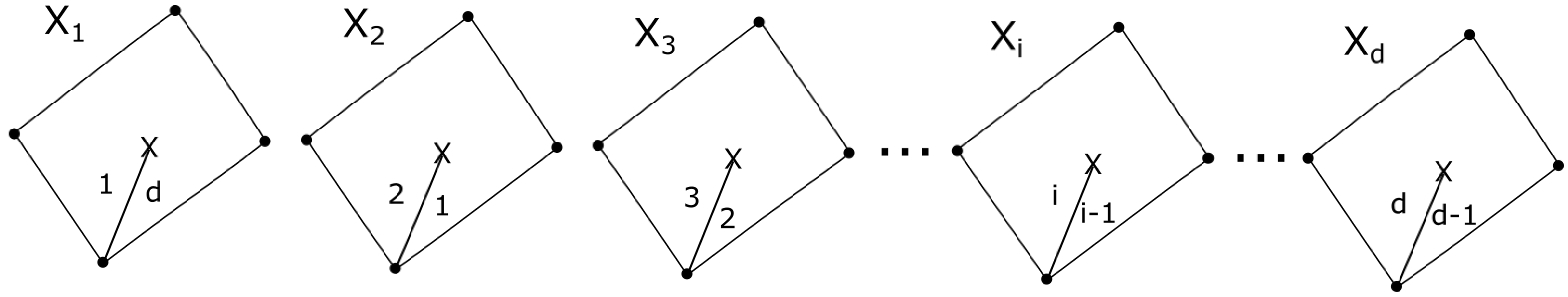
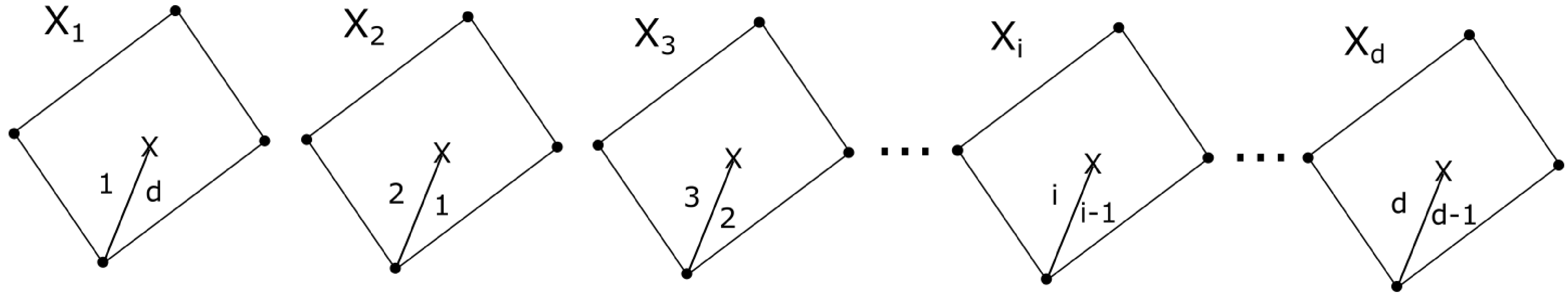
This is expected since doubled slit tori are not lattice surfaces.

Higher genus



These surfaces are called *symmetric torus covers*.

Higher genus

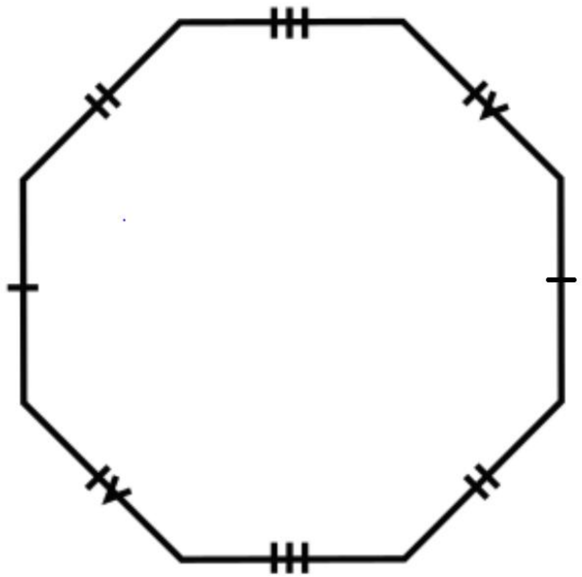


These surfaces are called *symmetric torus covers*.

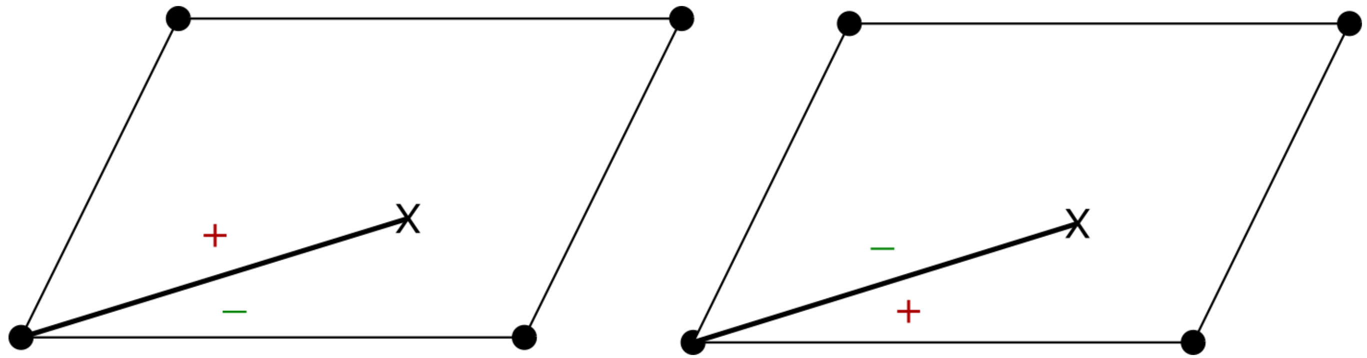
Symmetric torus covers have the same gap distribution as doubled slit tori.

Other results on gaps of translation surfaces

- Lattice surfaces (highly symmetric translation surfaces)

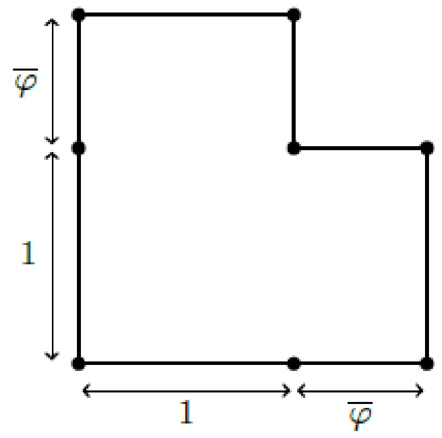


- Non-lattice surfaces



Gaps of lattice surfaces

- Athreya-Cheung (2014) - Torus
- Athreya-Chaika-Lelievre (2015) - Golden L
- Uyanik-Work (2016) - Regular octagon
- Taha (2020)- Gluing two regular (2n+1)-gons



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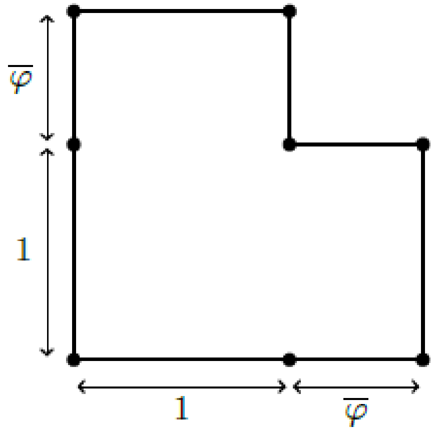
Characteristics of the gap distributions:

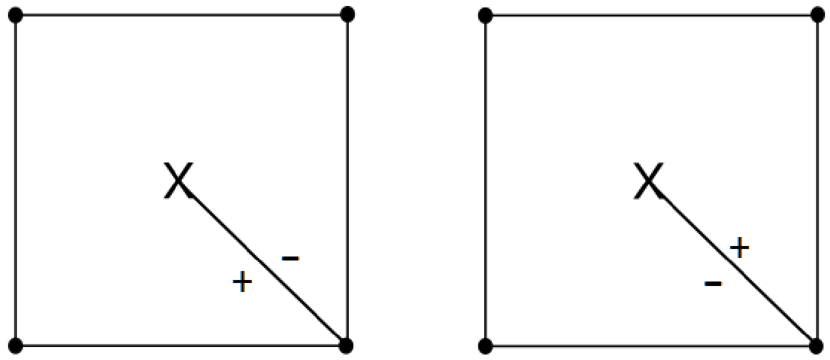
- No small gaps
- 2-dimensional parameter space
- Explicit gap distributions



Gaps of non-lattice surfaces

Athreya-Chaika (2012) – Generic translation surfaces

- Gap distribution exists for a.e. translation surface and is the same
- Non-explicit
- Small gaps characterize non-lattice surfaces



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Work (2019) – $\mathcal{H}(2)$ Genus 2, single cone point

- Parameter space 6-dimensional
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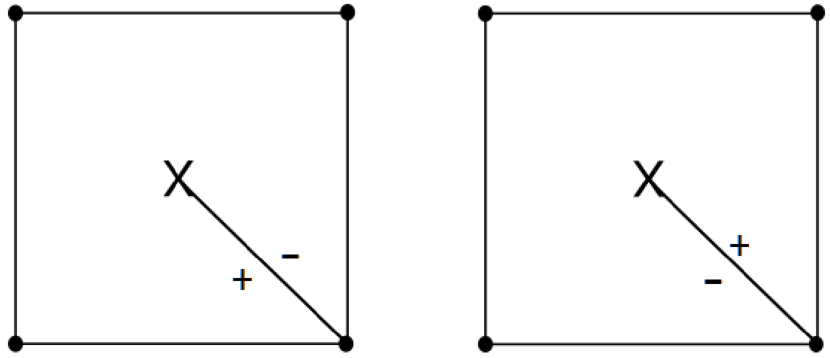

Gaps of non-lattice surfaces

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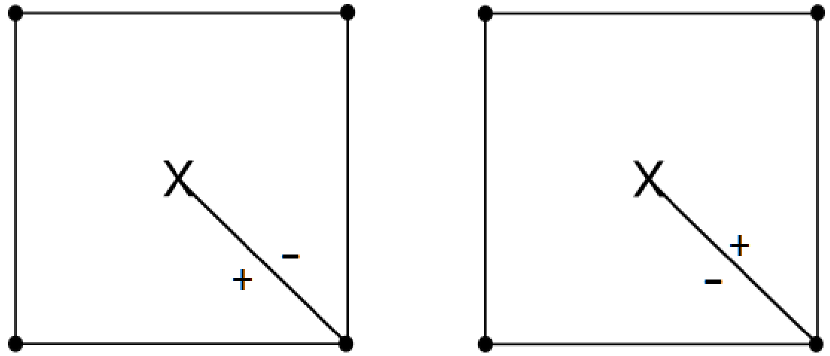
S. (2020) – Doubled slit tori

- Parameter space 4-dimensional
- First explicit gap distribution for non-lattice surface



This concludes Part 1

Part 2: Elements of proof

Anthony Sanchez
asanch33@uw.edu

May 14th, 2020

Elements of the proof

- Turn gap question into a dynamical question
- On return times and affine lattices

Guiding philosophy

Questions about a *fixed* translation surface can be understood by considering the dynamics on the space of *all* translation surfaces.

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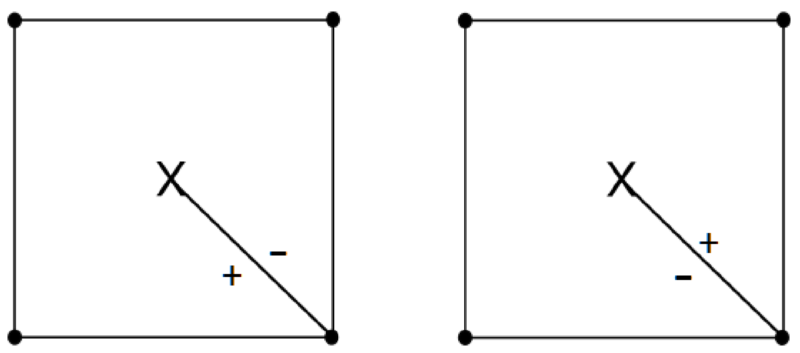
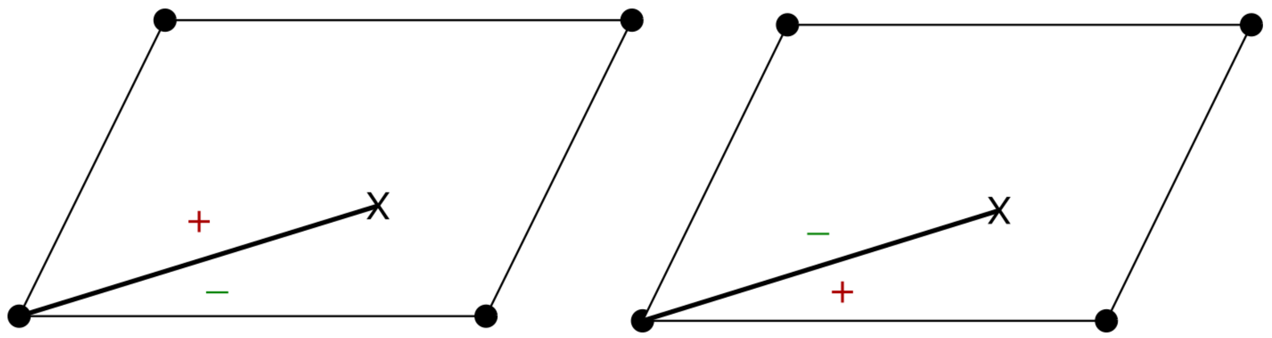
Questions about a *fixed* translation surface can be understood by considering the dynamics on the space of *all* translation surfaces.

Gap distribution of a doubled slit torus → Dynamical question on the space of doubled slit tori

Translation surfaces $\mathcal{E}$

Let $\mathcal{E}$ denote the set of all doubled slit tori





The $SL(2, \mathbb{R})$-action

There is a “linear” action of $SL(2, \mathbb{R})$ on $\mathcal{E}$

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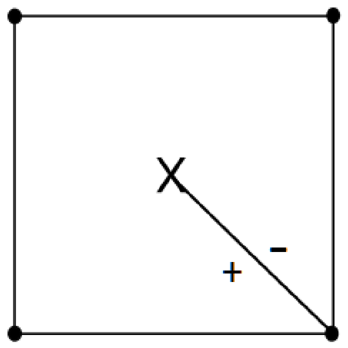

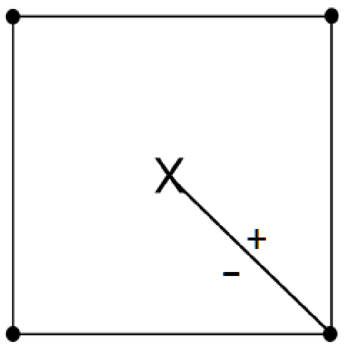

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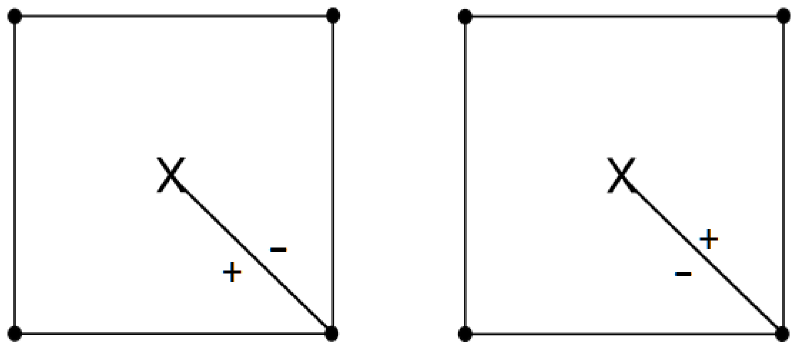

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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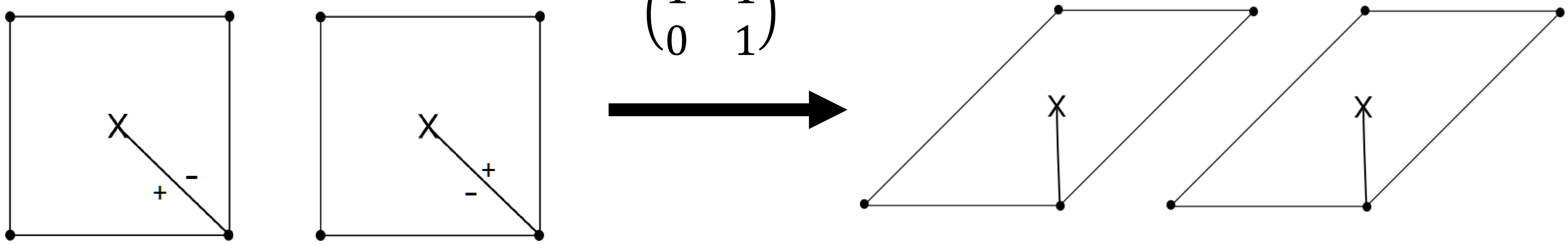

Horocycle flow

Consider the 1-parameter family

$$\left\{ h_u = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} : u \in \mathbb{R} \right\}$$

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Horocycle flow

Consider the 1-parameter family

$$\left\{ h_u = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix} : u \in \mathbb{R} \right\}$$

- Vertical shear on the plane.
- This subgroup is of interest because of how it changes slopes.

Slopes

$$h_u\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y-ux\end{pmatrix}$$

Slopes

$$h_u \binom{x}{y} = \binom{x}{y - ux}$$

$$\downarrow$$

$$slope\left(h_u \binom{x}{y}\right)$$

Slopes

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$$\Downarrow$$

$$slope\left(h_u\begin{pmatrix}x\\y\end{pmatrix}\right) = slope\left(\begin{pmatrix}x\\y\end{pmatrix}\right) - u$$

Slopes

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$$\Downarrow$$

$$slope\left(h_u \binom{x}{y}\right) = slope\left(\binom{x}{y}\right) - u$$

In particular, ***slope differences*** are preserved!

Transversal for doubled slit tori

Consider the ***transversal*** for doubled slit tori

$$\mathcal{W} = \{\omega \epsilon \mathcal{E} | \Lambda_\omega \cap (0,1] \neq \emptyset\}$$

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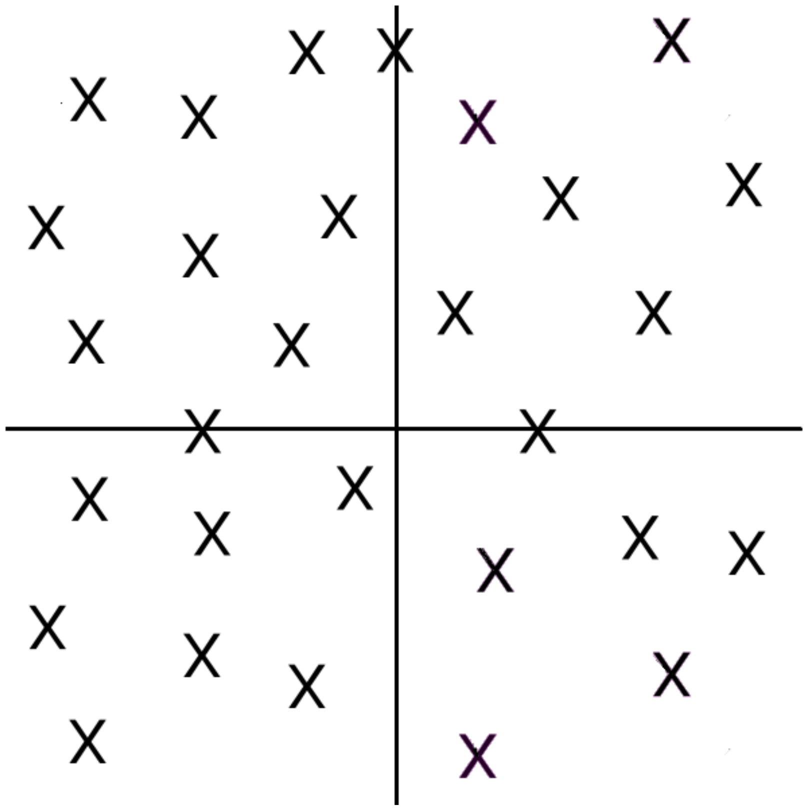
That is, the doubled slit tori that have a *short* horizontal saddle connection.

Transversal for doubled slit tori

Λ_ω



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Key: slope gaps = return times to $\mathcal{W}$

- **First return time**:

If $\omega \epsilon \mathcal{W}$, when is $h_u \omega \epsilon \mathcal{W}$?

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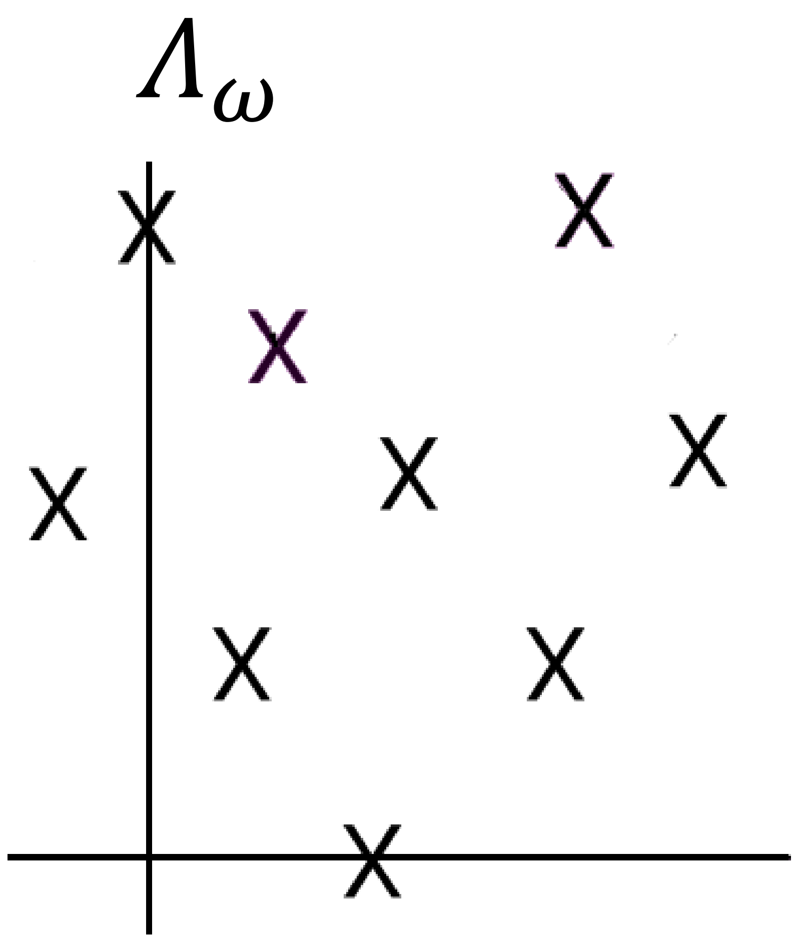

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If $\omega \epsilon \mathcal{W}$, when is $h_u \omega \epsilon \mathcal{W}$?

Need a vector in Λ_ω with

$$h_u \binom{x}{y} = \binom{x}{y - ux}$$

short and horizontal.

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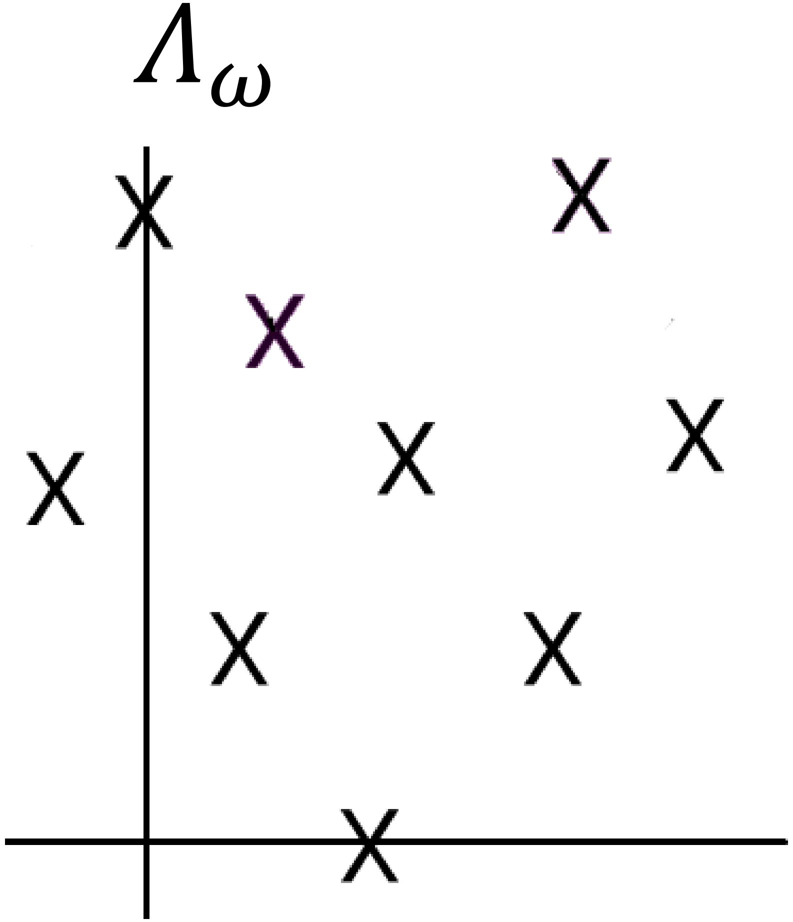

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Need a vector in Λ_ω with

$$h_u \binom{x}{y} = \binom{x}{y - ux}$$

short and horizontal.

- This happens is when

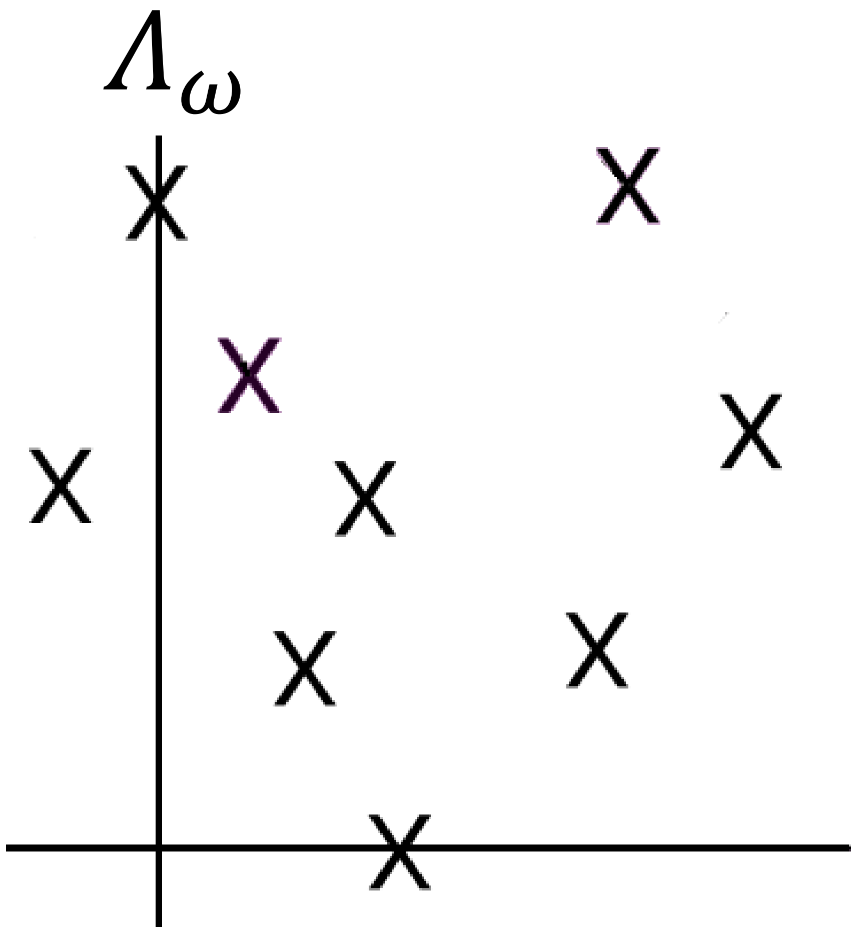
$$y - ux = 0 \Leftrightarrow u = \frac{y}{x}$$

So the ***first return time*** is a slope

So the ***first return time*** is a slope

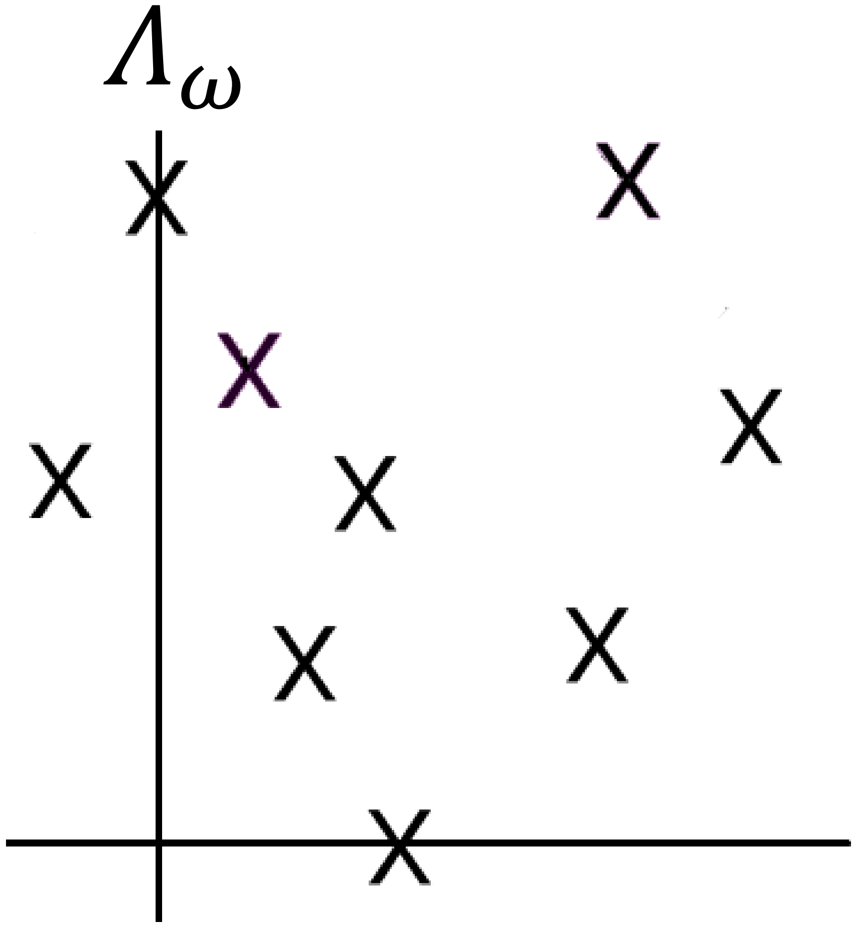
What about the second return time?

Second return time



Second return time = total time minus the first return time

Second return time



Second return time = total time minus the first return time

Hence, second return time is a slope difference.

Formalizing the key idea

Let R denote the *return time*

Let T denote the *return map*

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$$R(\omega) = \inf\{u > 0 | h_u(\omega) \in \mathcal{W}\}$$

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Formalizing the key idea

slope gaps = return times to $\mathcal{W}$

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↓

$$s_{i+1} - s_i = R\left(T^i(\omega)\right)$$

Slope gaps as a dynamical question

$$\frac{|Gaps^N(\Lambda_\omega) \cap I|}{N}$$

Slope gaps as a dynamical question

$$\frac{|Gaps^N(\Lambda_\omega) \cap I|}{N} = \frac{1}{N} \sum_{i=0}^{N-1} \chi_{\{R^{-1}(I)\}}(T^i(\omega))$$

Slope gaps as a dynamical question

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Slope gaps as a dynamical question

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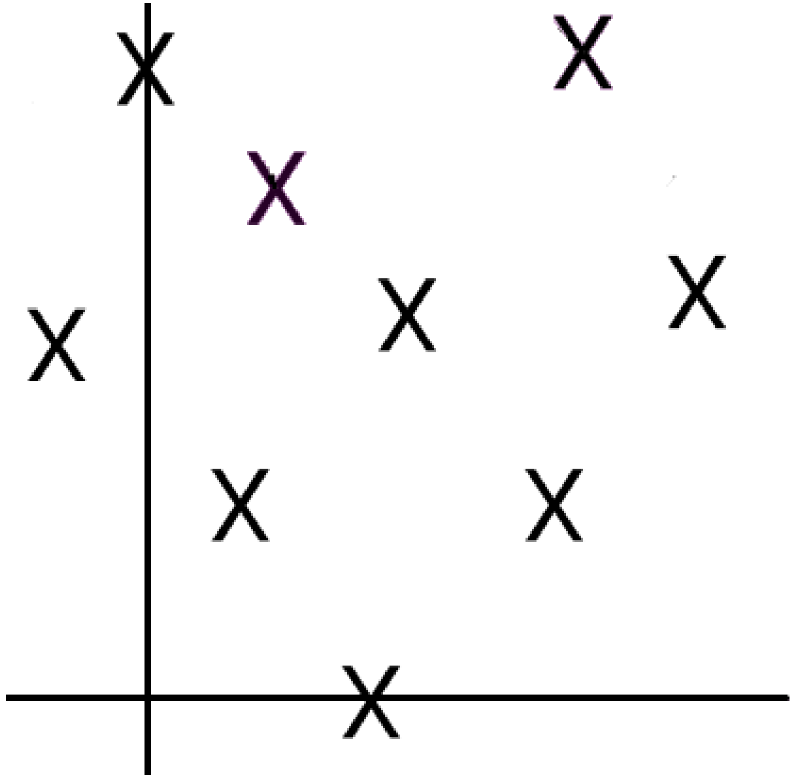
$$\to \mu\{\omega \epsilon \mathcal{W} \,|R(\omega) \in I\}$$

So next steps:

- parametrize $\mathcal{W}$
- find return map in coordinates

Part 2: Finding the return time

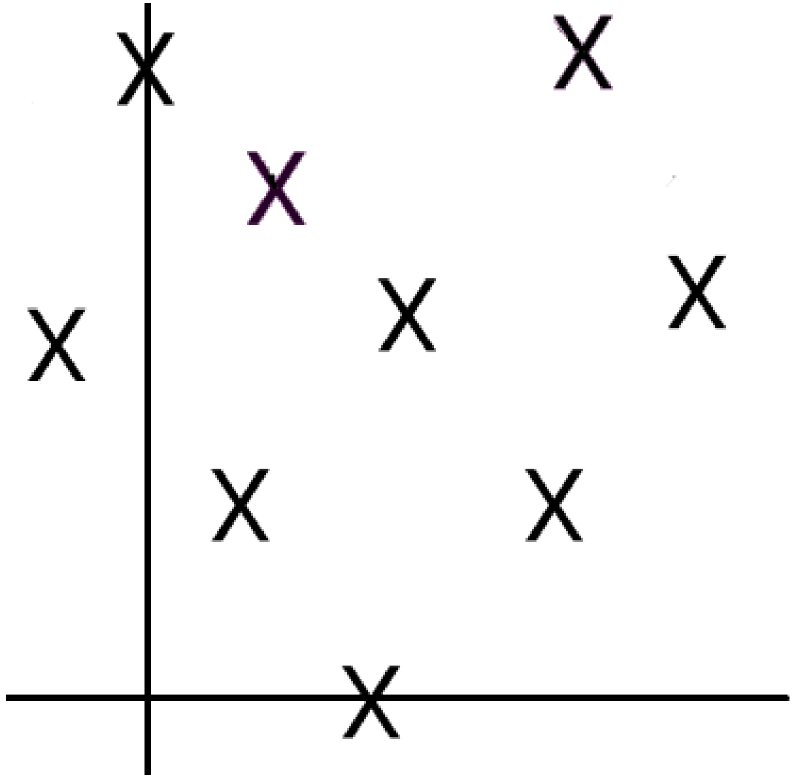
Return time = slope of the next vector to become short



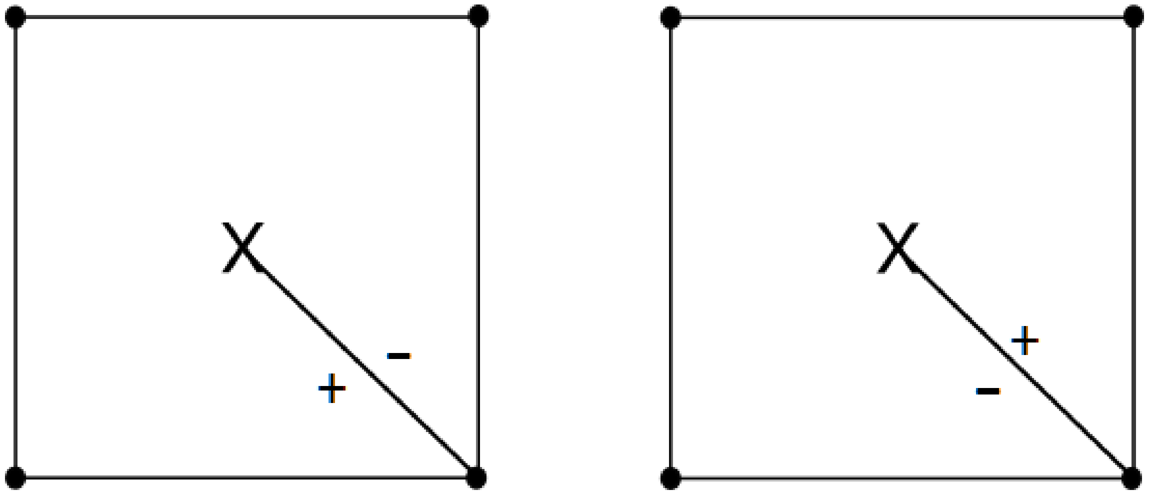
Part 2: Finding the return time

Return time = slope of the next vector to become short

The rest of the talk we will only concern ourselves with vectors of smallest positive slope



Understanding saddle connections



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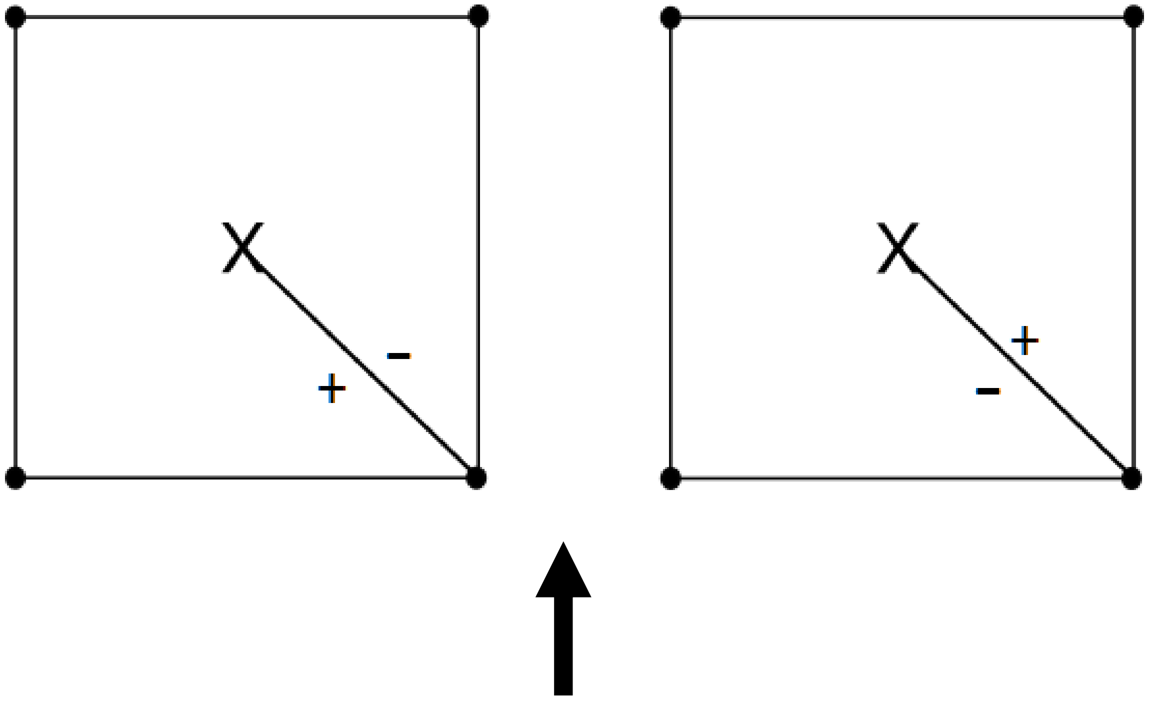
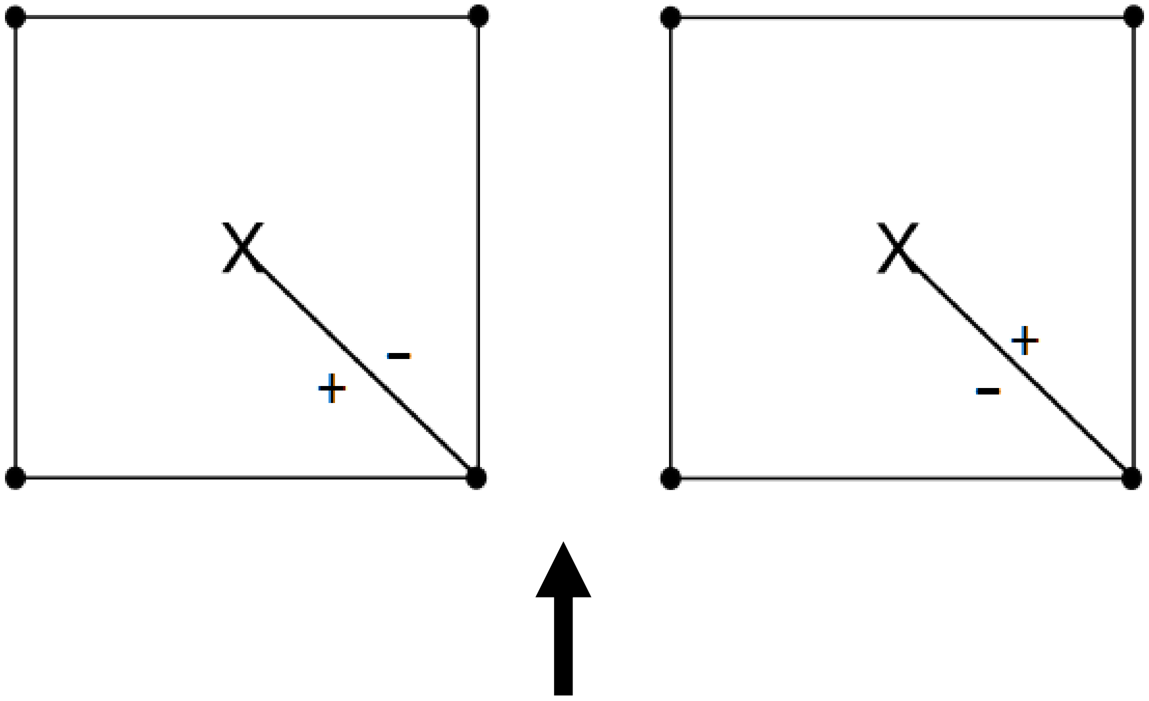


$$\mathbb{C}/_{\mathbb{Z}^2}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Understanding saddle connections



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Two types of saddle connections

- $\mathbb{Z}^2$

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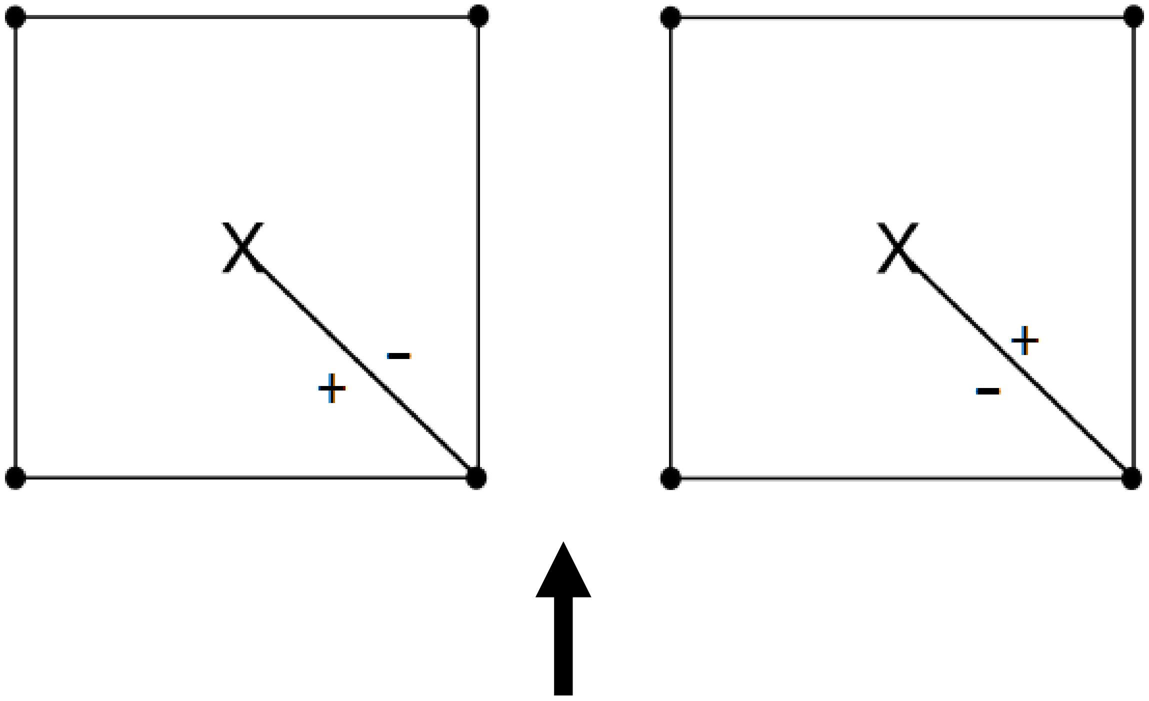
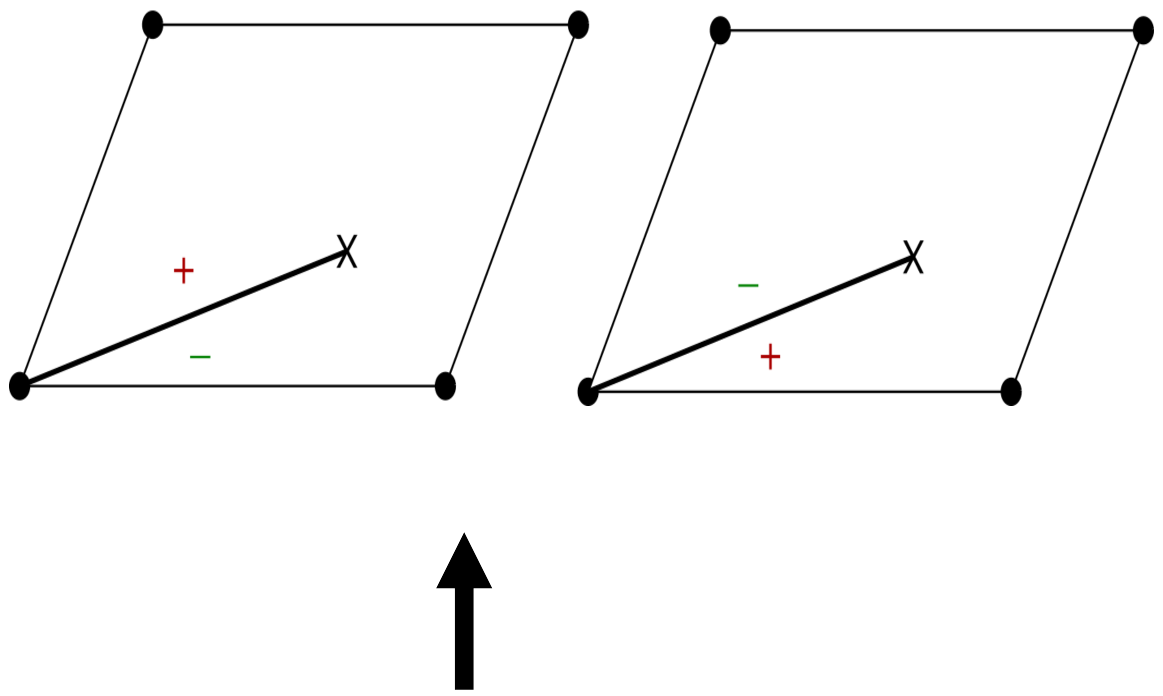


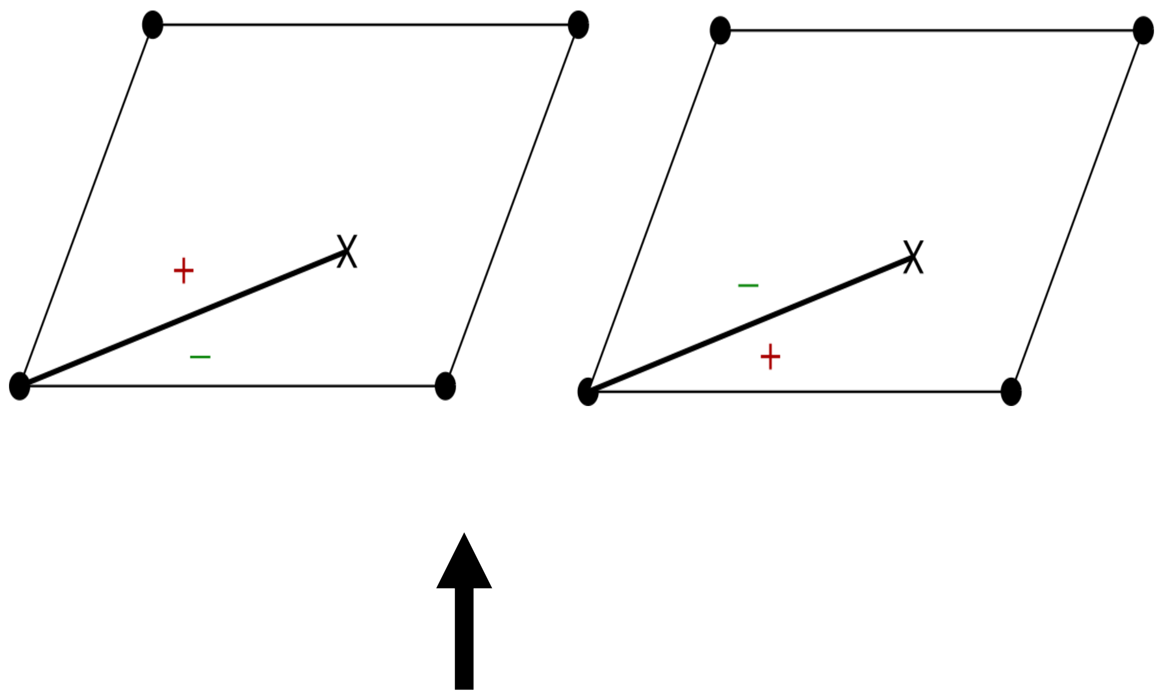
$$\mathbb{C}/_{\mathbb{Z}^2}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Two types of saddle connections

- $\mathbb{Z}^2$
- $\mathbb{Z}^2 + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$



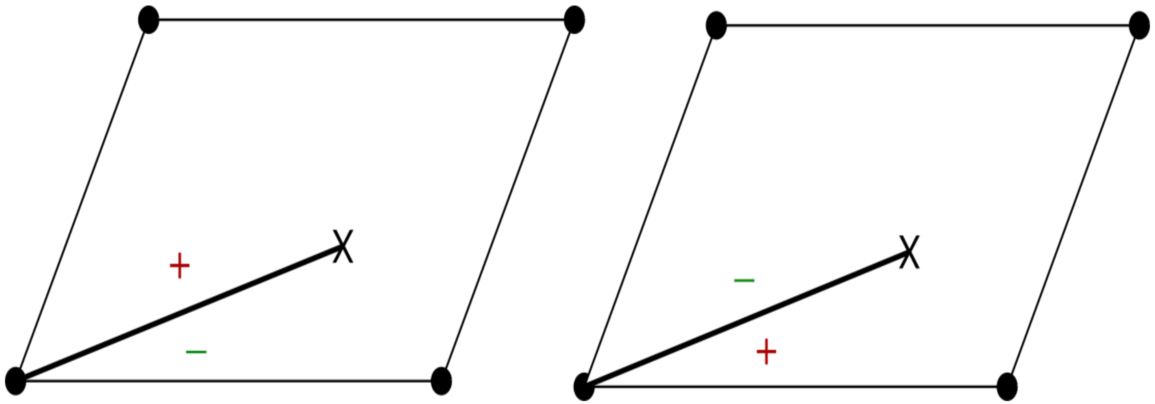
$$\mathbb{C}/_{g\mathbb{Z}^2}, v$$



$$\mathbb{C}/_{g\mathbb{Z}^2}, v$$

Two types of saddle connections

- $g\mathbb{Z}^2$

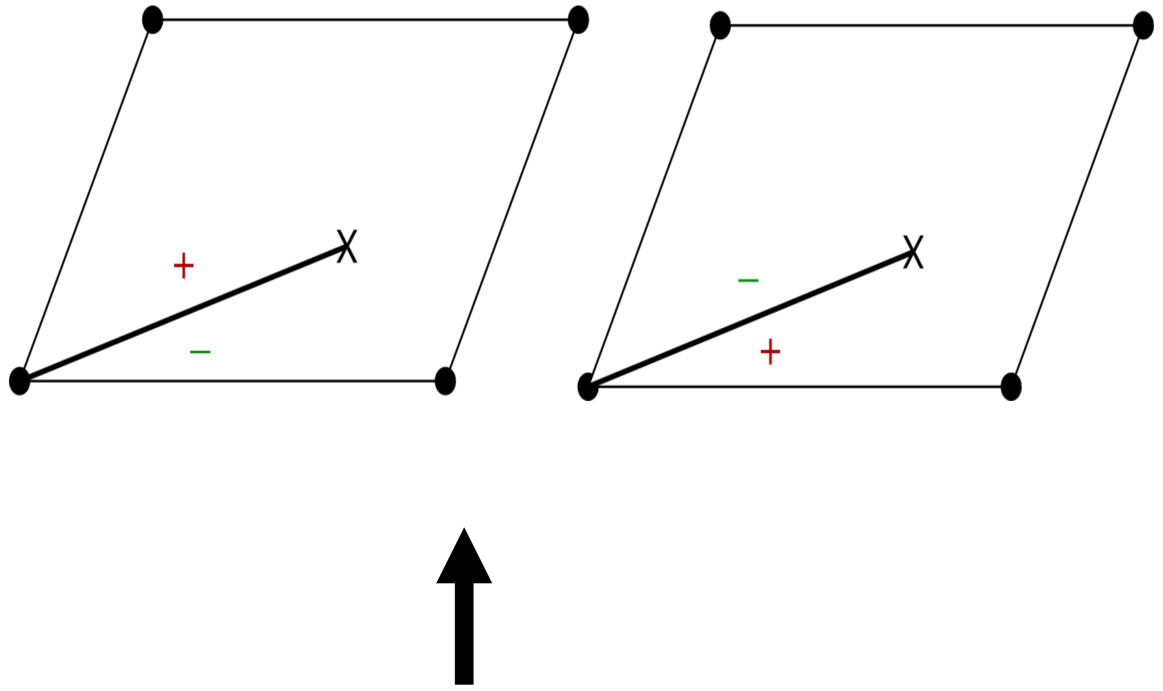


$\mathbb{C}/_{g\mathbb{Z}^2}, v$

Two types of saddle connections

- $g\mathbb{Z}^2$

Understood by torus results



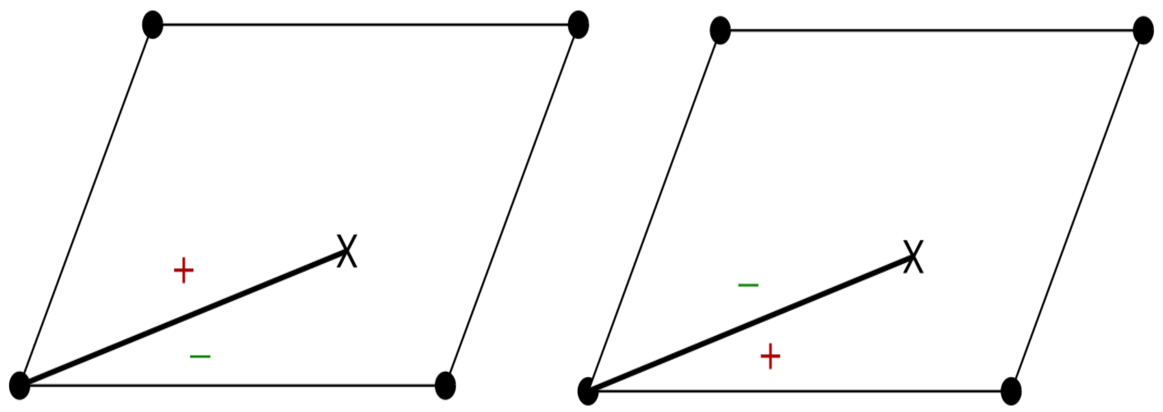
$\mathbb{C}/_{g\mathbb{Z}^2}, v$

Two types of saddle connections

- $g\mathbb{Z}^2$

Understood by torus results

- $g\mathbb{Z}^2 + v$



$\mathbb{C}/_{g\mathbb{Z}^2}, v$

Two types of saddle connections

- $g\mathbb{Z}^2$

Understood by torus results

- $g\mathbb{Z}^2 + v$

Defines an affine lattice!

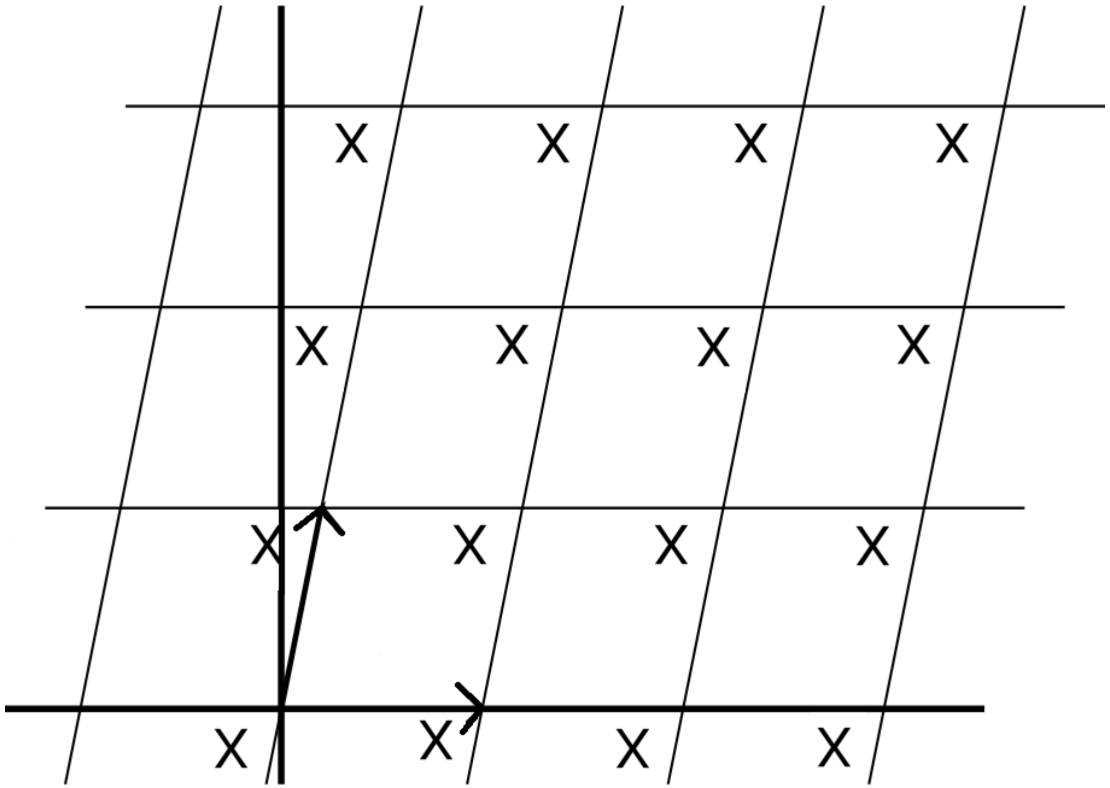
Parameterizing affine lattices

Data needed for an **affine lattice**

$\Lambda = g\mathbb{Z}^2 + v$ is

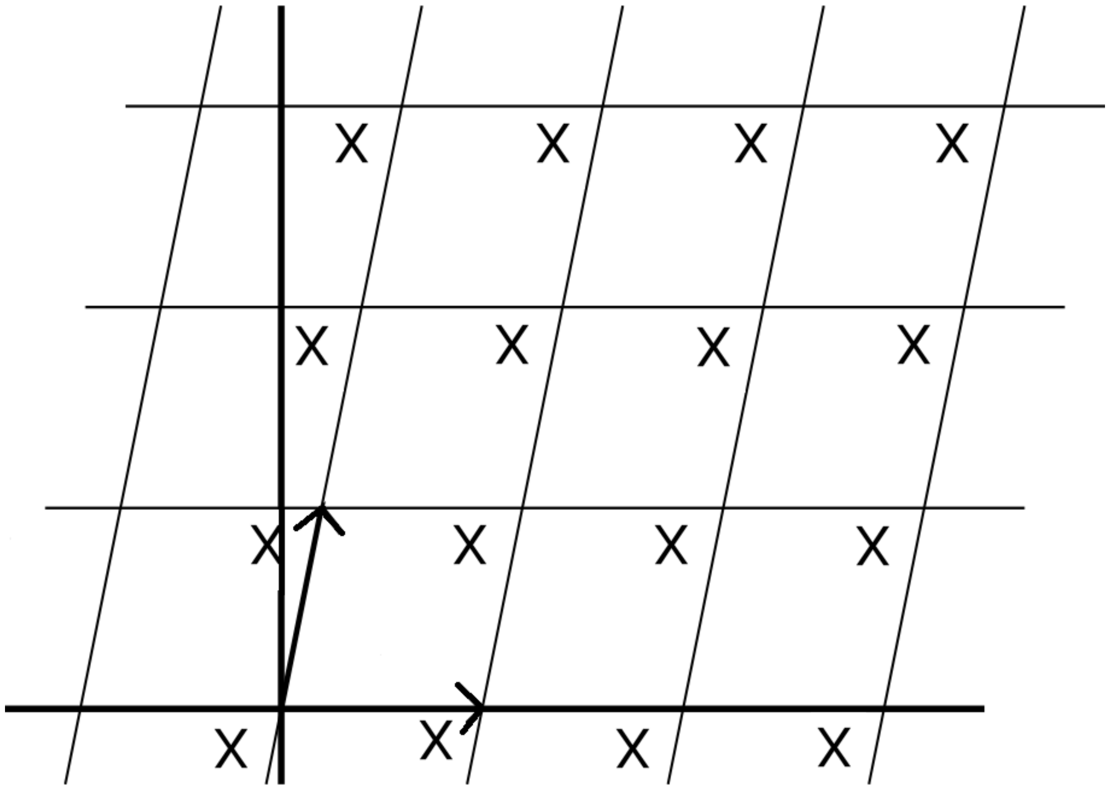
- lattice $g \in SL(2, \mathbb{R})$
- vector $v \in \mathbb{C}/g\,\mathbb{Z}^2$

$$\Lambda = g\mathbb{Z}^2 + v.$$



$$\Lambda = g\mathbb{Z}^2 + v.$$

Given an affine lattice $\Lambda = g\mathbb{Z}^2 + v$, what is the short vector of smallest slope?

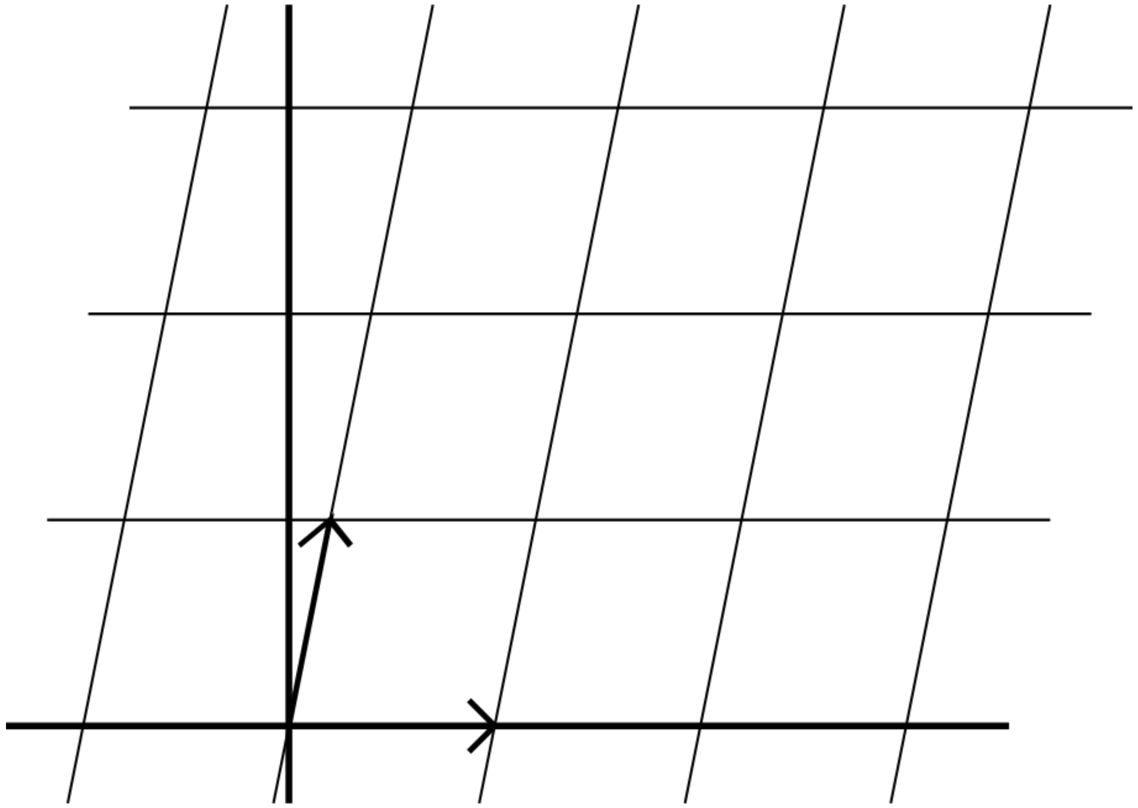


A special case

Consider the affine lattices of the form

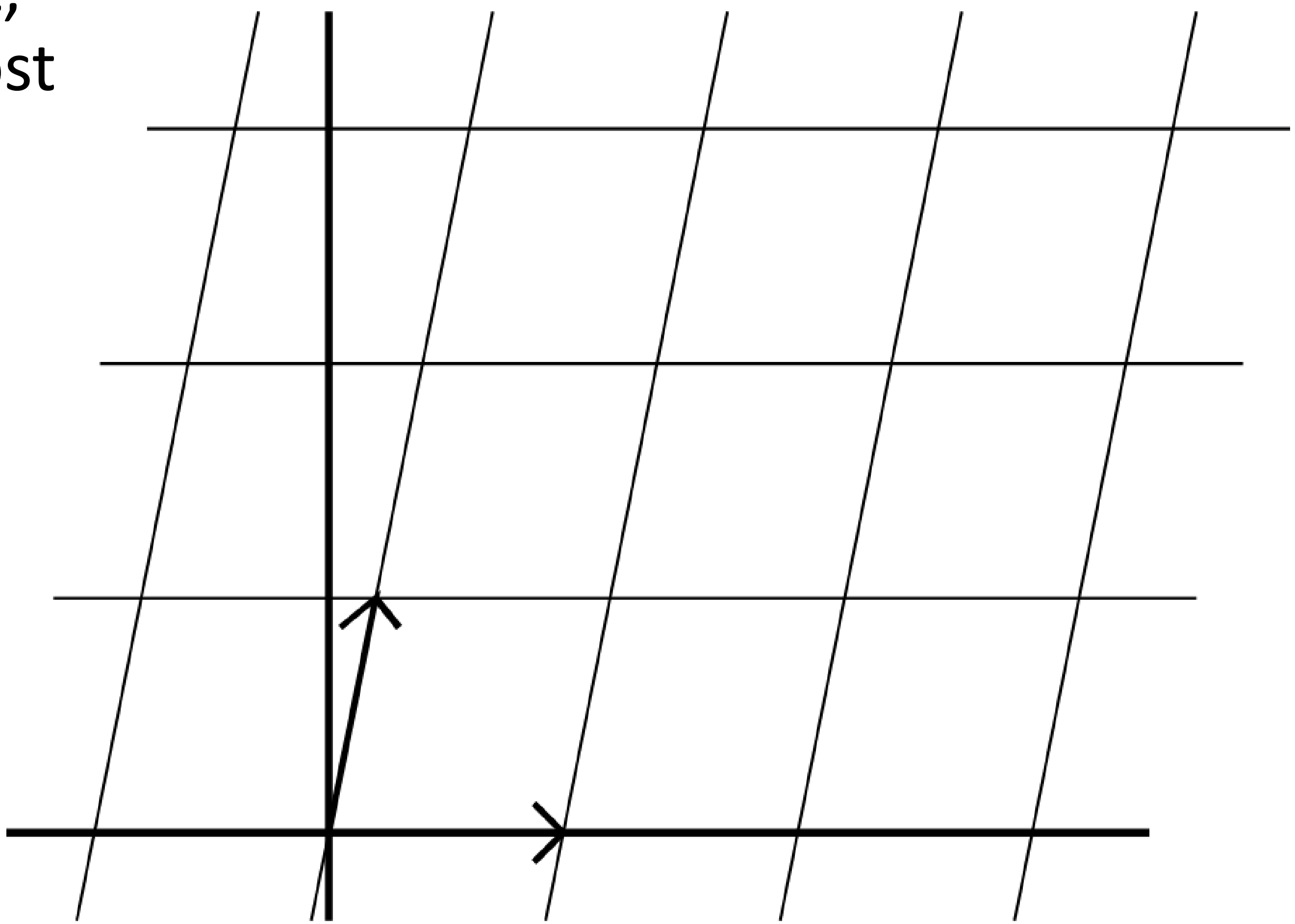
$$\Lambda = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

What are the vectors of smallest slope?



$$\Lambda = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

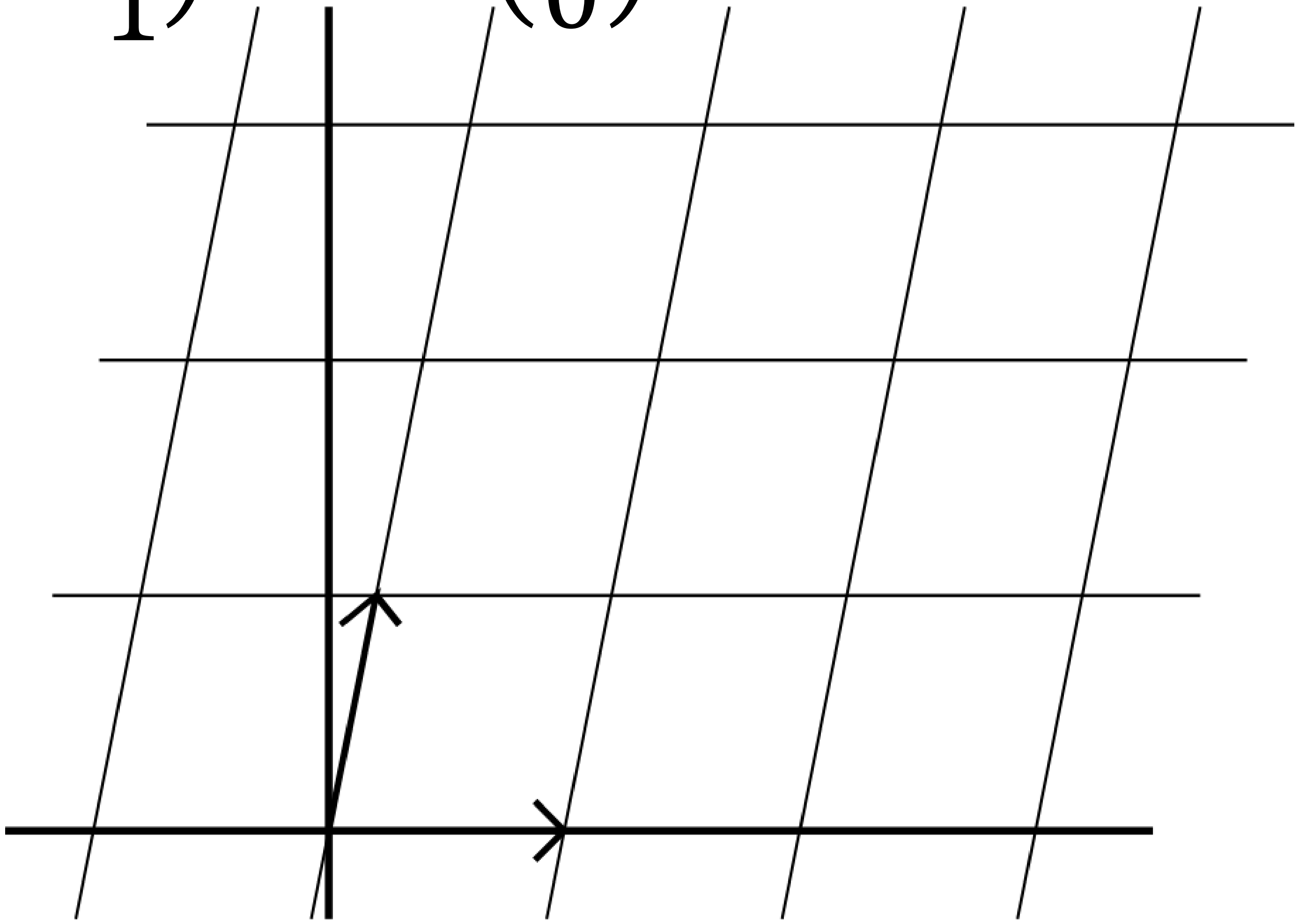
At every height, can have at most one vector in a unit length interval.



Strategy for $\Lambda = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

So to find vector of smallest non-zero slope

- Consider the affine vector $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$.
- Use structure of the lattice and track how slope changes



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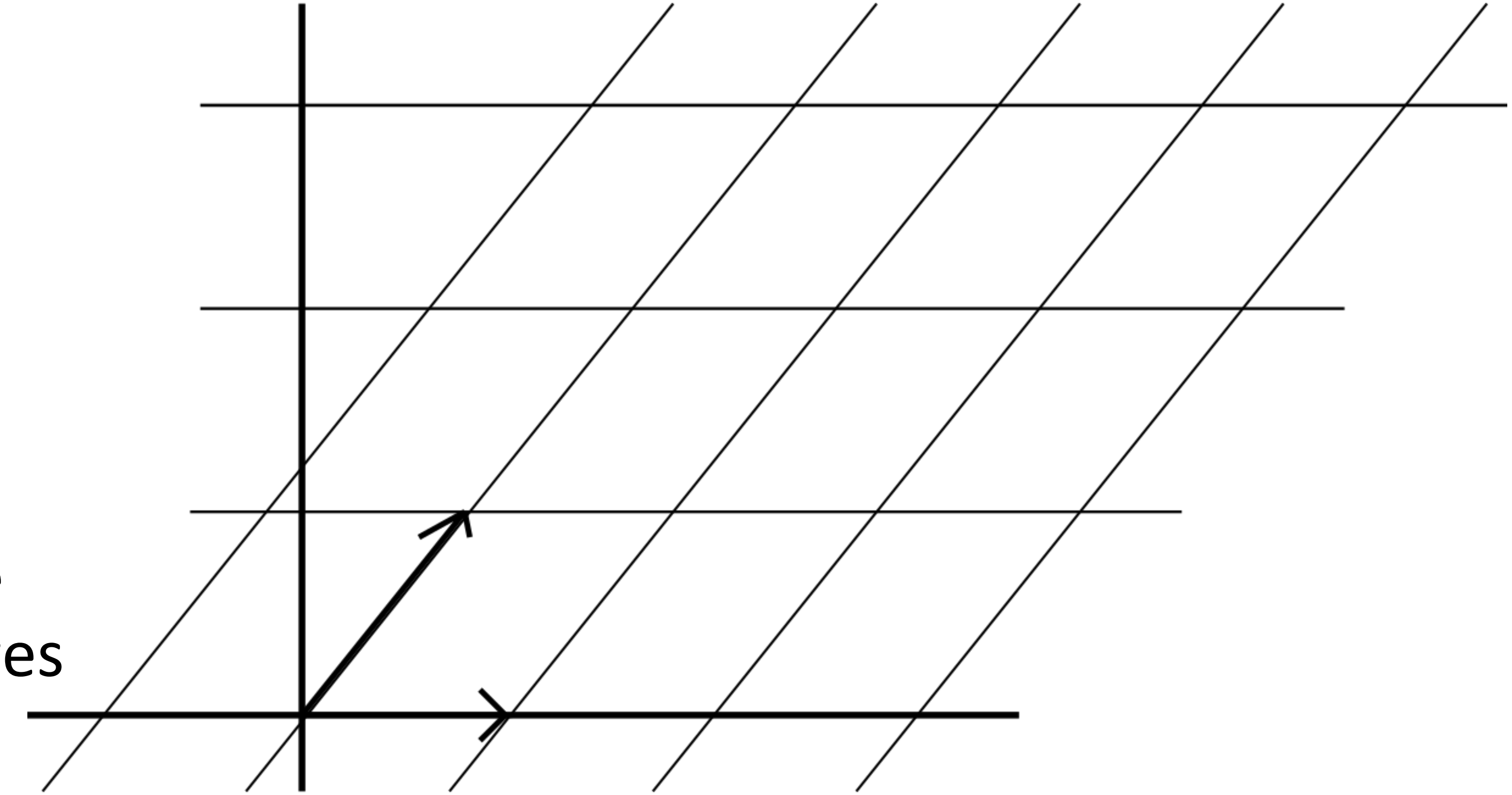

Short vectors of $\Lambda = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

The next vector to become short

$$\begin{cases} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \text{second basis vector}, & if\ b + \alpha < 1 \\ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} - \text{first basis vector + (many) second basis}, & if\ b + \alpha > 1 \end{cases}$$

Short vectors of $\Lambda = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$

The next vector to become short

$$\begin{cases} \begin{pmatrix} b+\alpha \\ 1 \end{pmatrix}, & if\ b+\alpha < 1 \\ \begin{pmatrix} jb+\alpha-1 \\ \text{j} \end{pmatrix}, & if\ b+\alpha > 1 \end{cases}$$

where $j = \left\lfloor \frac{2-\alpha}{b} \right\rfloor$

Elements of the proof

- This idea (with some modifications) is used to find holonomy vectors of doubled slit tori of smallest slope

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- These are the return times to the transversal

Elements of the proof

- This idea (with some modifications) is used to find holonomy vectors of doubled slit tori of smallest slope
- These are the return times to the transversal
- This answer answers the gap distribution question for doubled slit tori

Thank you!

Special thanks to:

- Dr. Jayadev Athreya (My advisor)
- West Coast Dynamics Seminar