EXPANDING GRAPHS AND PROPERTY (T)

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1. Expanders

1.1. **Definitions and analysis on graphs.** Let $G = (V, E)$ be a (possibly infinite) graph. We allow self-loops and multiple edges. For $x \in V$ the *neighbourhood of* x is the multiset $N_x = \{y \in V | (x, y) \in E\}$. Let $E(A, B) = |E \cap A \times B|$, $e(A, B) = |E(A, B)|$, $e(A) = e(A, V)$ for $A, B \subseteq V$. We G is *locally finite*, i.e. that $d_x = |N_x|$ is finite for all $x \in V$. We will consider the space $L^2(V)$ under the measure $\mu(\{x\}) = d_x$. Note that $e(V)$ is *twice* the (usual) number of edges in the graph.

Definition 1.1. The "local average" operator $A : L^2(V) \to L^2(V)$ of G is:

$$(Af)(x) = \frac{1}{d_x} \sum_{y \in N_x} f(y).$$

It is a self-adjoint operator on $L^2(\mu)$ since

$$\langle Af, g \rangle_V = \sum_{x \in V} d_x \left(\frac{1}{d_x} \sum_{y \in N_x} f(y) \right) \overline{g(x)} = \sum_{y \in V} d_y f(y) \frac{1}{d_y} \sum_{x \in N_y} \overline{g(x)} = \langle f, Ag \rangle_V .$$

Two applications of Cauchy-Schwarz give

$$|\langle Af, g \rangle_V| \leq \sum_{v \in V} d_v |f(v)| \frac{1}{d_v} \sum_{u \in N_v} |g(u)| \leq \sum_{v \in V} |f(v)| \sqrt{d_v} \left(\sum_{u \in N_v} |g(u)|^2 \right)^{1/2}$$
$$\leq \left(\sum_{v \in V} |f(v)|^2 d_v \right)^{1/2} \left(\sum_{v \in V} \sum_{u \in N_v} |g(v)|^2 \right)^{1/2} = \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)} ,$$

which means $\|A\|_{L^2(\mu)} \leq 1$.

From now on we assume that G has finite components. Then by the maximum principle, $Af = f$ iff f is constant on connected components of G and $Af = -f$ iff f takes opposing values on the two sides of each bipartite component.

Definition 1.2. The *discrete Laplacian* on V is the opeartor $\Delta = I - A$.

By the previous discussion it is self-adjoint, positive definite and of norm at most 2. The kernel of Δ is spanned by the characteristic functions of the components (e.g. if G is connected then zero is a non-degenerate eigenvalue). Its orthogonal complement $L^2_0(V)$ is the space of *balanced* functions (i.e. the ones who average to zero on each component of G). The *spectral gap* $\lambda_1(G)$ (the infimum of the positive eigenvalues) is an important parameter. If $\lambda_1(G) \geq \lambda$ we call G a λ*-expander*.

Definition 1.3. Let $A \subset V$. The *boundary of* A is $\partial A = E(A, \neg A)$. The *Cheeger constant* of the graph G is:

$$h(G) = \min \left\{ \frac{e(A, \neg A)}{e(A, V)} \Big| A \subseteq V,\ e(A \cap X) \leq \frac{1}{2} e(X) \text{ for every component } X \subseteq V \right\}.$$

Proposition 1.4. $h(G) \geq \frac{\lambda_1}{2}$.

Proof. Let X be a component, $A \subset X$ such that $2e(A) \le e(X)$, let $B = X \setminus A$, and choose α, β so that $f(x) = \alpha 1_A(x) + \beta 1_B(x)$ is balanced. Then we have: $\lambda_1(G) \le \frac{\langle \Delta f, f\rangle_V}{\langle f, f\rangle_V}$. Now,

$$\Delta f(x) = \begin{cases} \alpha - \frac{|N_x \cap A|}{|N_x|}\alpha - \frac{|N_x \cap B|}{|N_x|}\beta & x \in A \\ \beta - \frac{|N_x \cap A|}{|N_x|}\alpha - \frac{|N_x \cap B|}{|N_x|}\beta & x \in B \end{cases} = \begin{cases} \frac{|N_x \cap B|}{|N_x|}(\alpha - \beta) & x \in A \\ \frac{|N_x \cap A|}{|N_x|}(\beta - \alpha) & x \in B \end{cases}$$

so that $\langle \Delta f, f\rangle_V = (\alpha - \beta)\alpha|\partial A| + \beta(\beta - \alpha)|\partial B| = (\alpha - \beta)^2|\partial A|$ and thus

$$\lambda_1(G) \le \frac{(1 - \frac{\beta}{\alpha})^2}{e(A) + e(B)(\beta/\alpha)^2}|\partial A|.$$

$\langle f, \mathbb{1}_X\rangle_V = e(A)\alpha + e(B)\beta$, so that the choice $\beta/\alpha = -e(A)/e(B)$ makes f balanced. This means:

$$\lambda_1(G) \le |\partial A|\frac{(e(B) + e(A))^2}{e(A)e(B)^2 + e(B)e(A)^2} = 2\frac{|\partial A|}{e(A)}\frac{e(B) + e(A)}{2e(B)}.$$

But $2e(B) \ge e(X) = e(A) + e(B)$ and we are done. □

Conversely,

Proposition 1.5. $h(G) \le \sqrt{2\lambda_1(G)}$.

Proof. Let f be an eigenfunction of Δ of e.v. $\lambda \le \lambda_1 + \varepsilon$, w.l.g. supported on a component X and everywhere real-valued. Let $A = \{x \in V | f(x) > 0\}$, $B = X \setminus A$. We can assume $e(A) \le \frac{1}{2}e(X)$ by taking $-f$ instead of f if necessary. Let $g(x) = \mathbb{1}_A(x)f(x)$. Then for $x \in A$,

$$\begin{aligned} \Delta f(x) = f(x) - \frac{1}{d_x}\sum_{y \in N_x} f(x) &= g(x) - \frac{1}{d_x}\sum_{y \in N_x \cap A} f(x) - \frac{1}{d_x}\sum_{y \in N_x \cap B} f(x) \\ &= \Delta g(x) + \frac{1}{d_x}\sum_{y \in N_x \cap B}(-f(x)) \ge \Delta g(x). \end{aligned}$$

Since also $(\Delta f)(x) = \lambda f(x)$ for all x, we have:

$$\lambda \sum_{x \in A} d_x g(x)^2 = \sum_{x \in A} d_x \Delta f(x) \cdot g(x) \ge \sum_{x \in A} d_x \Delta g(x) \cdot g(x),$$

or ($g\restriction_B = 0$):

$$\lambda_1 + \varepsilon \ge \lambda \ge \frac{\langle \Delta g, g\rangle_V}{\langle g, g\rangle_V}.$$

we now estimate $\langle \Delta g, g\rangle_V$ in a different fashion. Motivated by the continuous fact: $\nabla g^2 = 2g\nabla g$, we evaluate

$$I = \sum_{x \in V} d_x \frac{1}{d_x}\sum_{y \in N_x} |g(x)^2 - g(y)^2|$$

in two different ways. On the one hand,

$$I = \sum_{(x,y) \in E} |g(x) + g(y)| \cdot |g(x) - g(y)| \le \left(\sum_{(x,y) \in E}(g(x) + g(y))^2\right)^{1/2}\left(\sum_{(x,y) \in E}(g(x) - g(y))^2\right)^{1/2},$$

and we note that

$$\begin{aligned} \sum_{(x,y) \in E}(g(x) - g(y))^2 &= \sum_{x \in V} d_x g(x)\frac{1}{d_x}\sum_{y \in N_x}(g(x) - g(y)) - \sum_{y \in V} d_y g(y)\frac{1}{d_x}\sum_{x \in N_y}(g(x) - g(y)) \\ &= 2\langle \Delta g, g\rangle_V \end{aligned}$$

and

$$\sum_{(x,y)\in E}(g(x)+g(y))^2 \le 2\sum_{(x,y)\in E}(g(x)^2+g(y)^2) = 4\langle g,g\rangle_V\,,$$

so:

$$I^2 \le 8\langle \Delta g, g\rangle_V \cdot \langle g,g\rangle_V \le 8\lambda_1 \langle g,g\rangle_V^2\,. \tag{1.1}$$

On the other hand, let $g(x)$ take the values $\{\beta_i\}_{i=0}^r$ where $0=\beta_0<\beta_1<\cdots<\beta_r$, and let $L_i=\{x\in V|g(x)\ge\beta_i\}$ (e.g. $L_0=V$). Then write:

$$I = 2\sum_{(x,y)\in E}\;\sum_{a(x,y)<i\le b(x,y)}(\beta_i^2-\beta_{i-1}^2)$$

where $\{\beta_{a(x,y)},\beta_{b(x,y)}\}=\{g(x),g(y)\}$ (i.e. replace $\beta_b^2-\beta_a^2$ with $(\beta_b^2-\beta_{b-1}^2)+\cdots+(\beta_{a+1}^2-\beta_a^2)$). Then the difference $\beta_i^2-\beta_{i-1}^2$ appears for every pair $(x,y)\in E$ such that $a(x,y)<i\le b(x,y)$ or such that $\max\{g(x),g(y)\}\ge\beta_i^2$ while $\min\{g(x),g(y)\}<\beta_i^2$. This exactly means than $(x,y)\in\partial L_i$ and

$$I = 2\sum_{i=1}^r\left(\beta_i^2-\beta_{i-1}^2\right)|\partial L_i|\,.$$

By definition of h, $L_i\subseteq A$ and $e(A)\le E$ imply $|\partial L_i|\ge h\cdot e(L_i)$ so:

$$I \ge 2h\sum_{i=1}^r\left(\beta_i^2-\beta_{i-1}^2\right)e(L_i) = 2h\sum_{i=1}^{r-1}\beta_i^2(e(L_i)-e(L_{i+1})) + 2h\cdot e(L_r)\beta_r^2.$$

Also, $e(L_i)-e(L_{i+1})=e(L_i\setminus L_{i+1})$ so:

$$I \ge 2h\sum_{i=1}^{r-1}\sum_{g(x)=\beta_i}\beta_i^2 d_x + 2h\cdot\sum_{g(x)=\beta_r}\beta_r^2 d_x = 2h\sum_{x\in V}d_x g(x)^2 = 2h\cdot\langle g,g\rangle_V\,. \tag{1.2}$$

We now combine Equations 1.1 and 1.2 to get:

$$2h\langle g,g\rangle_V \le I \le 2\sqrt{2(\lambda_1+\varepsilon)}\,\langle g,g\rangle_V$$

for all $\varepsilon>0$, or

$$h(G)\le\sqrt{2\lambda_1(G)}.$$

□

Let us restate the previous two propositions in:

$$\frac{1}{2}\lambda_1(G)\le h(G)\le\sqrt{2\lambda_1(G)}.$$

1.2. **References, examples and applications.** The above propositions can be found in [2], with slightly different conventions. We also modify their definitions to read:

Definition 1.6. Say that G is an h_0-*expander* if $h(G)\ge h_0$. Say that G is a λ-*expander* if $\lambda_1(G)\ge\lambda$.

The previous section showed that both these notions are in some sense equivalent. Being well-connected, sparse (in particular regular) expanders are very useful, e.g. for sorting networks of finite depth (see), de-randomization (see),

The existence of expanders can be easily demonstrated by probabilistic arguments (see [3]). Infinite families of expanders are not difficult to find, e.g. the incidence graphs of $\mathbb{P}^1(\mathbb{F}_q)$ have $\lambda = 1-\frac{\sqrt{q}}{q+1}$ (as computed in [6] and later in [1]). However families of *regular* expanders are more difficult. The next section discusses the generalization by Alon and Milman in [2] of a construction due to Margulis [11]. For a different explicit family of regular expanders which enjoys additional useful properties see [10].

We remark here that there exists a bound for the asymptotic expansion constant of a family of expanders:

Theorem 1.7. *(Alon-Boppana) For every $\varepsilon > 0$ there exists $C = C(k, \varepsilon) > 0$ such that if G is a connected k-regular graph on n vertices, the number of eigenvalues of A in the interval*

$$[(2-\varepsilon)\frac{\sqrt{k-1}}{k}, 1]$$

is at least $C \cdot n$.

Corollary 1.8. *Let $\{G_m\}_{m=1}^{\infty}$ be a family of connected k-regular graphs such that $|V_m| \to \infty$. Then*

$$\limsup_{m\to\infty} \lambda_1(G_m) \leq 1 - \frac{2\sqrt{k-1}}{k}.$$

This leads to the following definition (the terminlogy is justified by [10]):

Definition 1.9. A k-regular graph G such that $|\lambda| \leq 2\frac{\sqrt{k-1}}{k}$ for every eigenvalue $\lambda \neq \pm 1$ of A is called a *Ramanujan graph*.

1.3. **Cayley graphs and property (T).** One way of generating families of finite regular graphs is by taking quotients of groups. Let Γ be a discrete group, and let $S \subset \Gamma$ be finite symmetric (i.e. $\gamma \in S \iff \gamma^{-1} \in S$) not containing the identity. Then for any subgroup $N < \Gamma$ of finite index, we can construct a finite graph $\mathrm{Cay}(N\backslash\Gamma; S)$ as follows: the vertices will be the right N-cosets $N\backslash\Gamma$, and we will take an edge (x, xs) for any coset $x = N\gamma$ and any $s \in S$. Note that if S actually generates Γ then $\mathrm{Cay}(N\backslash\Gamma; S)$ is connected for all N.

Clearly $G = \mathrm{Cay}(N\backslash\Gamma; S)$ is an $|S|$-regular graph. Furthermore, the set of vertices comes naturally equipped with the Γ action of right translation (which is not an action on the *graph* unless Γ is Abelian). This makes $L^2_0(V)$ into a unitary Γ-representation with no Γ-fixed vectors (these would be constant!). Now let $A\dot{\cup}B = V(G)$ and consider the balanced function $f(x) = b1_A(x) - a1_B$ where $a = |A|, b = |B|$. Then $\|f\|_2^2 = \frac{1}{|S|}b^2a + a^2b = \frac{abn}{|S|}$. It is easy to see that:

$$|(sf)(x) - f(x)| = \begin{cases} a+b & x \in A, xs \in B \text{ or } x \in B, xs \in A \\ 0 & x, xs \text{ both in } A \text{ or } B \end{cases}.$$

Thus we have:

$$\|sf - f\|_2^2 = \frac{1}{|S|}n^2|\partial_s A|.$$

Where we write $\partial_s A = \{(x,y) \in E(A,B) | y = sx \vee y = s^{-1}x\}$ so that $\partial A = \cup_{s\in S}\partial_s A$ and we only need to take one of every pair $s, s^{-1} \in S$. Clearly to insure that ∂A is large it suffices to make $\partial_s A$ large for some $s \in S$. It is thus natural to consider groups Γ of the following type:

Definition 1.10. (see Lemma 2.20) Let Γ be a discrete group, $S \subset \Gamma$ a finite subset. Then Γ has *property (T) with Kazhdan constant* $\varepsilon > 0$ *w.r.t.* S if for any unitary representation $\rho : \Gamma \to \mathrm{Aut}(\mathscr{H})$ of Γ such that $\mathscr{H}$ has no nontrivial Γ-fixed vectors, and any $x \in \mathscr{H}$ of norm 1 there exists $s \in S$ such that $1 - \langle \rho(s)x, x\rangle_{\mathscr{H}} \geq \varepsilon$. The largest ε for which this holds is called <u>the</u> *Kazhdan constant* of (Γ, S).

Note that then $\|\rho(s)x - x\|^2_{\mathscr{H}} = 2 - 2\langle \rho(s)x, x\rangle_{\mathscr{H}} \geq 2\varepsilon$.

Corollary 1.11. *(Alon-Milman* [2]*) If Γ has property (T) w.r.t. a symmetric generating subset S then for every $N \lhd \Gamma$ of finite index, Cay$(\Gamma/N; S)$ is an $\frac{\varepsilon}{|S|}$-expander.*

Proof. Let $A \subset \Gamma/N$ and assume $|A| \leq \frac{1}{2}n$ (note that a Cayley graph is regular). Let $f \in L^2_0(\Gamma/N)$ be as above. Then for some $s \in S$, $\|sf - f\|_2 \geq \varepsilon \|f\|_2$ and therefore ($2b \geq n$ by assumption!)

$$\frac{|\partial A|}{e(A)} \geq \frac{|\partial_s A|}{|A||S|} \geq 2\varepsilon\frac{1}{an^2} \cdot \frac{abn}{|S|} = \frac{\varepsilon}{|S|}\frac{2b}{n} \geq \frac{\varepsilon}{|S|}$$

∎

2. Property (T)

2.1. **The Fell Topology and its properties.** Let G be a locally compact group, and let $\tilde{G}$ (resp. $\hat{G}$) be the set of equivalence classes of unitary representations[1] (resp. irreducible unitary representations) of G. A basis of open neighbourhoods for the *Fell topology* (see [5][2] from which the following discussion is taken) on $\tilde{G}$ is the sets $U(\rho, \{v_i\}_{i=1}^r, K, \varepsilon)$ defined for each $(\rho, V) \in \tilde{G}$, an finite subset $\{v_i\}_{i=1}^r \subset V$ of vectors of norm 1, a compact $K \subseteq G$ and some $\varepsilon > 0$ by:

$$U(\rho, \{v_i\}_{i=1}^r, K, \varepsilon) = \left\{ (\sigma, W) \in \tilde{G} \middle| \exists \{w_j\}_{j=1}^r \subset W : \|w_j\|_W = 1 \wedge \forall g \in K \forall i,j : \left| \langle \rho(g) v_i, v_j \rangle_V - \langle \sigma(g) w_i, w_j \rangle_W \right| < \varepsilon \right\}.$$

This forms a basis for a topology since if $(\sigma, W) \in U(\rho, \{v_i\}, K, \varepsilon)$ let $\{w_j\} \subset W$ be of norm 1 as in the definition. K is compact, so

$$\delta = \varepsilon - \max_{i,j} \left\| \langle \rho(g) v_i, v_j \rangle_V - \langle \sigma(g) w_i, w_j \rangle_W \right\|_{L^\infty(K)}$$

is positive and then $U(\sigma, w, K, \delta) \subseteq U(\rho, v, K, \varepsilon)$. Also in this spirit we have:

Proposition 2.1. *Let $f : G \to H$ be a continuous homomorphism of groups. Let $f^* : \tilde{H} \to \tilde{G}$ be the pull-back map of representation. Then f^* is continuous in the Fell topology.*

Proof. Let $(\rho, V) \in \tilde{H}$, $\{v_i\} \in V$, $K \subset G$ be compact and $\varepsilon > 0$. Then $f^{*-1}(U_{\tilde{G}}(f^*\rho, \{v_i\}, K, \varepsilon)) \supseteq U_{\tilde{H}}(\rho, \{v_i\}, f(K), \varepsilon)$. □

Corollary 2.2. *If $H < G$ then $\mathrm{Res}_H^G : \tilde{G} \to \tilde{H}$ is continuous, since it is dual to the inclusion map of H in G.*

Corollary 2.3. *Assume that $f(G)$ is dense in H. Since the Fell topology of $\hat{G}$ is its induced topology as a subset of $\tilde{G}$, and since the pull-back of an irreducible H-representation is in this case irreducible as a G-representation, one can replace $\tilde{G}, \tilde{H}$ with $\hat{G}, \hat{H}$ in the previous proposition and corollary.*

Example 2.4. If G is abelian, then the Fell topology on $\hat{G}$ coincides with the Pontryagin topology.

Example 2.5. As in the abelian case, if G is compact then $\hat{G}$ is discrete:

Proof. Note that in this case we can always take $K = G$ in the definition above. Let $(\rho, V), (\sigma, W) \in \hat{G}$ and Consider the operators $T_V = \int_G \overline{\chi_\rho(g)} \rho(g) dg$ and $T_W = \int_G \overline{\chi_\rho(g)} \sigma(g) dg$ acting on V, W respectively. They commute with the respective representation since χ_ρ is a class function, $\chi_\rho(hg) = \chi_\rho(h^{-1}(hg)h) = \chi_\rho(gh)$. So by Schur's lemma they act by scalars. Note that $\mathrm{Tr}\, T_W = \int_G \overline{\chi_\rho(g)} \chi_\sigma(g) dg$ which is 1 if $\rho \simeq \sigma$ and 0 otherwise. Thus if $\rho \not\simeq \sigma$ we have $T_V = \frac{1}{\dim V}$ while $T_W = 0$. Now let $v \in V, w \in W$ be of norm 1, and consider

$$A = \langle T_V v, v \rangle_V - \langle T_W w, w \rangle_W = \frac{1}{\dim V}.$$

We also have:

$$A = \int_G \left| \overline{\chi_\rho(g)} \right| \left(\langle \rho(g) v, v \rangle_V - \langle \sigma(g) w, w \rangle_W \right) dg$$

and thus

$$|A| \le \| \langle \rho(g) v, v \rangle_V - \langle \sigma(g) w, w \rangle_W \|_{L^\infty(G)} \int_G \left| \overline{\chi_\rho(g)} \right| dg.$$

By the Cauchy-Schwarz inequality, $\int_G \left| \overline{\chi_\rho(g)} \right| dg \le \left(\int_G \left| \overline{\chi_\rho(g)} \right|^2 dg \right)^{1/2} \left(\int_G dg \right)^{1/2} = 1$ and thus for any representation σ distinct from ρ and any $w \in W$ we have:

$$\| \langle \rho(g) v, v \rangle_V - \langle \sigma(g) w, w \rangle_W \|_{L^\infty(G)} \ge \frac{1}{\dim V},$$

Which means that $U(\rho, v, G, \frac{1}{1+\dim V}) = \{(\rho, V)\}$ as desired. □

[1]For set-theoretic reasons, one should only consider representations on Hilbert spaces of bounded (large) cardinality.

[2]We are considering the "quotient topology" of that paper.

Lemma 2.6. *Let $H < G$ be closed. Then G/H is a separable locally compact Hausdorff space in the quotient topology.*

Definition 2.7. Let $H < G$ be closed, a Borel measure ρ on $H\backslash G$ is called *quasi-invariant* if $\rho(E) = 0 \iff \rho(Ex) = 0$ for any measurable $E \subset H\backslash G$ and any $x \in G$.

Let $H < G$ be closed, ρ a quasi-invariant Borel measure on G/H, and let $\lambda(x, y) = \frac{dR_y\rho}{d\rho}(x)$ be the Radon-Nikodym derivative where $R_y\rho(E) = \rho(Ey)$. This is a continuous function on G. Now let $(\pi, V) \in \tilde{H}$, and let

$$W' = \left\{ f \in M(G, V) \middle| \forall h \in H, x \in G : f(hx) = \pi(h)f(x) \right\}.$$

Note that if $f, g \in W'$ then $\langle f(hx), g(hx)\rangle_V = \langle \pi(h)f(x), \pi(h)g(x)\rangle_V = \langle f(x), g(x)\rangle_V$ so that $\langle f(x), g(x)\rangle_V$ is an H-invariant $\mathbb{C}$-valued function on G. In particular we can define

$$W = \left\{ f \in W' \middle| \int_{G/H} \|f(x)\|_V^2 \, d\rho(x) < \infty \right\}$$

and (identifying functions which are equal ρ-a.e.) we obtain a Hilbert space structure on W with the inner product $\langle f, g\rangle_W = \int_{H\backslash G} \langle f(x), g(x)\rangle_V \, d\mu(x)$. Completeness is a direct consequence of the completeness of V and standard arguments. Furthermore if $f \in W$ and $y \in G$ then $\sqrt{\lambda(x, y)}f(xy)$ (as a function of x) is also in W and has the same norm as f. We can thus define a representation of G on W by $(\sigma(g)f)(x) = \lambda(x, y)f(xy)$.

Definition 2.8. Let $H < G$ be closed. We call (σ, W) the representation of G *induced* by the representation (π, V) of H.

Lemma 2.9. *Let $H < G$ be closed. Then there exists a quasi-invariant Borel measure ρ on $H\backslash G$. Furthermore, if ρ_1, ρ_2 are two such measures then $(\rho_1, W_1) \simeq (\rho_2, W_2)$ as G-representations.*

Let dh be a right Haar measure on H, and let $\phi \in C_c(G, V)$ (norm topology on V). We can then define:

$$f_\phi(x) = \int_H \pi(h)\phi(h^{-1}x)dh$$

which is easily verified to be an element of W. Note that $f_\phi$is a continuous function on G with compact support $\mod H$ (i.e. $\|f_\phi(x)\|_V$ is of compact support on G/H).

Lemma 2.10. *The space of $\{f_\phi | \phi \in C_c(G, V)\}$ is dense in W.*

Also, if $\phi_n \to \phi$ uniformly on G, all of them supported on a single compact set, then $f_{\phi_n} \to f$ in the topology of W. We note that the subspace of $C_K(G, V)$ (elements of $C_c(G, V)$ supported on the compact K) generated by the functions of the form $\phi(x) = \alpha(x)v$ where $v \in V$ is fixed and $\alpha \in C_K(G, \mathbb{C})$ is dense in the sup-norm. We thus have:

Corollary 2.11. *The subspace $\mathcal{L} = \{\sum_{i=1}^n f_{\alpha_i v_i} | \alpha_i \in C_c(G, \mathbb{C}), v_i \in V, \xi_i \in \mathbb{C}\}$ is dense in W as well.*

Theorem 2.12. *If H is a closed subgroup of G then $\mathrm{Ind}_H^G : \tilde{H} \to \tilde{G}$ is continuous.*

Proof. Let $(\sigma, W) = \mathrm{Ind}_H^G(\pi, V) \in \tilde{G}$. Let $K \subseteq G$ be compact, $\{f_i\} \subset W$ be of norm 1 and $\varepsilon > 0$. We wish to prove that the inverse image of $U_{\tilde{G}}(\sigma, \{f_i\}, K, \varepsilon)$ contains an open neighbourhood of (π, V) in $\tilde{H}$. We first replace f_i by a "nicer" choice. The computation ($w_1, w_2, w_1', w_2' \in W$

are of norm 1):

$$\begin{aligned} |\langle\sigma(g)w_1, w_2\rangle_W - \langle\sigma(g)w_1', w_2'\rangle_W| &\le |\langle\sigma(g)w_1, w_2\rangle_W - \langle\sigma(g)w_1', w_2\rangle_W| + |\langle\sigma(g)w_1', w_2\rangle_W - \langle\sigma(g)w_1', w_2'\rangle_W| \\ &\le \|\sigma(g)w_1'\|_W \|w_2 - w_2'\|_W + \|w_2\|_W \|\sigma(g)(w_1 - w_1')\|_W = \|w_1 - w_1'\|_W + \|w_2 - w_2'\|_W . \end{aligned}$$

shows that if we replace f_i with $f'_i \in \mathcal{L}$ of norm 1 such that $\|f_i - f'_i\|_W \le \frac{\varepsilon}{3}$ then $U_{\tilde{G}}(\sigma, \{f_i\}, K, \varepsilon) \supseteq U_{\tilde{G}}(\sigma, \{f'_i\}, K, \frac{\varepsilon}{3})$. In other words, we can assume w.l.g. that $f_i \in \mathcal{L}$, and specifically that

$$f_i = \sum_{j=1}^{n} f_{\alpha_{ij} v_j}$$

where $\|v_j\|_V = 1$ and w.l.g. $\|\alpha_{ij}\|_\infty \le 1$ (the last by repeating some $\alpha_{ij} v_j$ if needed). Let $C_{ij} = \operatorname{supp} \alpha_{ij}$ and let $C = \{e\} \cup_{i,j} C_{ij}$ which is a compact subset of G, containing the support of f.

The idea of the proof is as follows: if we can identify $\{v'_k\} \subset V'$ in a neighbouring representation that transforms like the $\{v_j\}$ we can reconstruct an $f'_i \in \operatorname{Ind}_{\tilde{H}}^{\tilde{G}}(\pi', V')$ that transforms like f_i. In fact, let $M = H \cap CC^{-1}CKC^{-1}$ (a compact subsetp of H), and consider

$$U = U_{\tilde{H}}(\pi, \{v_j\}, M, \delta).$$

If $(\pi', V') \in U$, $\operatorname{Ind}_H^G(\pi', V') = (\sigma', W')$ we will prove that $(\sigma', W') \in U_{\tilde{G}}(\sigma, f, K, \varepsilon)$ if δ is small enough. By definition we can choose $\{v'_j\} \subset V'$ such that $\left| \langle \pi(h) v_j, v_k \rangle_V - \langle \pi'(h) v'_j, v'_k \rangle_{V'} \right| < \delta$ for all $h \in C$, j, k. We then let $f'_i = \sum_{j=1}^{n} f_{\alpha_{ij} v'_j} \in W'$, so that

$$(\sigma(g) f_i)(x) = \sqrt{\lambda(x,g)} f_i(xg) = \sqrt{\lambda(x,g)} \sum_{j=1}^{n} \int_H \alpha_{ij}(h^{-1} x g) \pi(h) v_j \, dh$$

and

$$\langle \sigma(g) f_{i_1}, f_{i_2} \rangle_W = \int_{G/H} \langle (\sigma(g) f_{i_1})(x), f_{i_2}(x) \rangle_V \, d\rho(x)$$

$$= \sum_{j_1, j_2 = 1}^{n} \int_{G/H} \sqrt{\lambda(x,g)} d\rho(x) \int_{H \times H} \alpha_{i_1 j_1}(h_1^{-1} x g) \overline{\alpha_{i_2 j_2}(h_2^{-1} x)} \langle \pi(h_1) v_{j_1}, \pi(h_2) v_{j_2} \rangle_V \, dh_1 dh_2$$

$$= \sum_{j_1, j_2 = 1}^{n} \int_{G/H} d\rho(x) \sqrt{\lambda(x,g)} \int_H dh_2 \langle \pi(h_2) v_{j_1}, v_{j_2} \rangle_V \int_H dh_1 \alpha_{i_1 j_1}(h_1 x g) \overline{\alpha_{i_2 j_2}(h_2 h_1 x)}.$$

The same holds for σ' and f'_i so that:

$$\begin{aligned} \langle \sigma(g) f_{i_1}, f_{i_2} \rangle_W - \langle \sigma'(g) f'_{i_1}, f'_{i_2} \rangle_{W'} \quad = \quad & \sum_{j_1, j_2 = 1}^{n} \int_{G/H} d\rho(x) \sqrt{\lambda(x,g)} \\ & \int_H dh_2 \left(\langle \pi(h_2) v_{j_1}, v_{j_2} \rangle_V - \langle \pi'(h_2) v'_{j_1}, v'_{j_2} \rangle_{V'} \right) \\ & \int_H dh_1 \alpha_{i_1 j_1}(h_1 x g) \overline{\alpha_{i_2 j_2}(h_2 h_1 x)}. \end{aligned}$$

Now if $C \cap Hx = \emptyset$ then $\alpha_{i_2 j_2}(h_2 h_1 x) = 0$ for all i_2, j_2, h_1, h_2 and thus the inner integral is zero. In particular, the outer integral can be taken over the compact image $\bar{C}$ of C in G/H, and we may assume $x \in C$ in the inner integral. If furthermore $g \in K$ then $\alpha_{i_1 j_1}(h_1 x g) = 0$ unless $h_1 \in CK^{-1}C^{-1}$ (we need $h_1 x g \in C$) so we can take the inner integral over $H \cap CK^{-1}C^{-1}$, or:

$$\left| \int_H dh_1 \alpha_{i_1 j_1}(h_1 x g) \overline{\alpha_{i_2 j_2}(h_2 h_1 x)} \right| \le \mu_H(H \cap CK^{-1}C^{-1})$$

where $d\mu_H(h) = dh$. Secondly, if $x \in C$ and $h_1 \in CK^{-1}C^{-1}$ then $h_2 h_1 x \in C$ implies $h_2 \in CC^{-1}CKC^{-1}$ i.e. $h_2 \in M$. Thus h_2-integral can be taken over M instead, where

$$\left| \langle \pi(h_2) v_{j_1}, v_{j_2} \rangle_V - \langle \pi'(h_2) v'_{j_1}, v'_{j_2} \rangle_{V'} \right| \le \delta$$

so that

$$\left|\int_H dh_2 \left(\langle\pi(h_2)v_{j_1}, v_{j_2}\rangle_V - \langle\pi'(h_2)v'_{j_1}, v'_{j_2}\rangle_{V'}\right)\int_H dh_1 \alpha_{i_1j_1}(h_1xg)\overline{\alpha_{i_2j_2}(h_2h_1x)}\right| \le \mu_H(H\cap CK^{-1}C^{-1})\mu_H(M)\delta.$$

Since also

$$\int_{\bar{C}} d\rho(x)\sqrt{\lambda(x,g)} \le \int_{\bar{C}}(1+\lambda(x,g))d\rho(x) = \rho(\bar{C}) + \rho(\bar{C}g) \le \rho(\bar{C}) + \rho(\bar{C}K),$$

we finally have for all $g \in K$

$$\left|\langle\sigma(g)f_{i_1}, f_{i_2}\rangle_W - \langle\sigma'(g)f'_{i_1}, f'_{i_2}\rangle_{W'}\right| \le n^2(\rho(\bar{C}) + \rho(\bar{C}K))\mu_H(M)\mu_H(H\cap CK^{-1}C^{-1})\delta$$

and it is clear that $(\sigma', W') \in U_{\tilde{G}}(\sigma, \{f_i\}, K, \varepsilon)$ if δ is small enough. □

Remark 2.13. $U_{\hat{G}}(\pi, v, K, \varepsilon)$ (only one vector!) form a basis for the topology of $\hat{G}$.

Proof. Since these are open sets if suffices to prove that every $U_{\hat{G}}(\pi, \{v_i\}, K, \varepsilon)$ contains $U_{\hat{G}}(\pi, v, N, \delta)$ for some v, N, δ.

Take $v \in V_\pi$ of norm 1 such that $\{\sigma(g)v \mid g \in G\}$ span V_σ. Then there exist $T = \{t_j\}_{j=1}^r \subset H$ and $\{a_{ij}\} \subset \mathbb{C}$ are such that $\left\|v_i - \sum_j a_{ij}\pi(t_j)v\right\|_V < \delta$. Let $A = \max_i \sum_j |a_{ij}|^2$, $M = T^{-1}KT \cup T^{-1}T$ and let

$$U = U_{\hat{G}}(\pi, v, M, \delta).$$

We will prove that if δ is small enough then $(\pi', V') \in U$ implies $(\pi', V') \in U_{\hat{G}}(\pi, \{v_i\}, K, \varepsilon)$. By definition we can choose $v' \in V'$ such that $|\langle\pi(h)v, v\rangle_V - \langle\pi'(h)v', v'\rangle_{V'}| < A^{-1}\varepsilon$ for all $h \in N$. Now let $v'_i = \sum_j a_{ij}\pi'(t_j)v'$ and observe that since $t_{j_2}^{-1}gt_{j_1} \in N$ for all $1 \le j_1, j_2 \le r$, $g \in K$,

$$\left|\langle\pi(g)v_{i_1}, v_{i_2}\rangle_V - \langle\pi(g)v'_{i_1}, v'_{i_2}\rangle_{V'}\right| \le$$
$$\sum_{j_1,j_2}|a_{i_1j_1}a_{i_2j_2}|\left|\langle\pi(t_{j_2}^{-1}gt_{j_1})v, v\rangle_V - \langle\pi'(t_{j_2}^{-1}gt_{j_1})v', v'\rangle_{V'}\right| \le A\delta$$

by Cauchy-Schwarz. Setting $i_1 = i_2 = i$ and using $T^{-1}T \subset N$ shows that $\sqrt{1-A\delta} \le \|v'_i\|_V \le \sqrt{1+A\delta}$. The analysis at the beginning of the proof of the theorem then implies that for $v''_i = \hat{v}'_i$ then $\left|\langle\pi(g)v''_i, v''_j\rangle_{V'} - \langle\pi(g)v'_i, v'_j\rangle_{V'}\right| \le 2\sqrt{1+A\delta}\cdot\frac{A\delta}{1+\sqrt{1-A\delta}}$ and it is clear that for δ small enough we are done. □

2.2. **Kazhdan's Property (T).** This section is based on Kazhdan's paper [8], Chapter 3 of Lubotzky's book [9], as well as de la Harpe and Valette's book [4]

Definition 2.14. We say that the locally compact group G has *property (T)* if the trivial representation is an isolated point of $\hat{G}$ in the Fell topology.

Example 2.15. By example 2.5 every compact group has property (T). Using example 2.4 as well we find that an Abelian group has property (T) iff it is compact.

Example 2.16. Let G have property (T). Then every quotient H of G has property (T).

Proof. See Corollary 2.3. Note also that if $f(G)$ is dense in H then f^* maps non-trivial representations to non-trivial representations. □

Corollary 2.17. *Let G have property (T). Then $G^{ab} = G/[G,G]$ is compact, since it is an Abelian group with property (T).*

Definition 2.18. Let $\sigma, \rho \in \tilde{G}$. Say that σ is *contained* in ρ ($\sigma \in \rho$) if ρ has a subrepresentation isomorphic to σ, i.e. if there exists a G-equivariant Hilbert space embedding of the space of σ into the space of ρ.

We say that σ is *weakly contained* in ρ ($\sigma \propto \rho$) if $\sigma \in \overline{\{\rho\}}$ where the closure is in the topology of $\tilde{G}$. In other words, $\sigma \propto \rho$ iff every matrix element of σ is a uniform limit on compact sets of matrix elements of ρ.

Lemma 2.19. *G has property T iff $1 \propto \sigma$ implies $1 \in \sigma$ for every $\sigma \in \tilde{G}$.*

A stronger version is:

Lemma 2.20. *If G has property (T) then there exists an open neighbourhood $U = U_{\tilde{G}}(1, 1, K, \varepsilon')$ of the trivial representation such that if $\rho \in U$ then $1 \in \rho$.*

Proposition 2.21. *A group with property (T) is compactly generated.*

Proof. Assume G countable first, say $G = \{\gamma_n\}_{n=1}^{\infty}$, and let $H_n = \langle \gamma_1, \ldots, \gamma_n \rangle$. Then H_n is a closed subgroup of G, and consider the representation $\rho_n = \mathrm{Ind}_{H_n}^{G} 1$ (the L^2 functions on $H_n \backslash G$ w.r.t. counting measure with G acting by right translation). Since G isn't finitely generated, $H_n \backslash G$ is infinite, and thus there are no G-invariant vectors in ρ_n (i.e. the constant function on $H_n \backslash G$ isn't in L^2) and thus $1 \notin \rho = \hat{\oplus}_n \rho_n$. On the other hand for every compact (i.e. finite) subset $K \subset G$, there exists an n such that $K \subset H_n$ from some point onwards and then any unit vector in ρ_n is H_n-invariant, in particular K invariant so that $1 \propto \rho$.

For a general locally compact G, this reads as follows: for each compact subset $K \subset G$ let $H_K = \overline{\langle K \rangle}$ (the closed subgroup generated by K), and consider the representation $\rho_K = \mathrm{Ind}_{H_K}^{G} 1$. Note that any unit vector in ρ_K is K-invariant. If G isn't compactly generated, $H_K \backslash G$ not compact, hence of infinite quotient measure. In particular, there are no G-invariant vectors in ρ_K. Thus $1 \notin \rho = \hat{\oplus}_K \rho_K$. On the other hand ρ contains a K-invariant vector for every compact $K \subset G$ by construction, so that $1 \propto \rho$. □

The main result of Kazhdan's seminal paper[3] is:

Theorem 2.22. *Let G be a simple algebraic group defined over a local field F, of F-rank at least 2. Then G_F has property (T).*

This is useful for our purposes due to:

Proposition 2.23. *Let G be locally compact, $\Gamma < G$ be a closed subgroup such that there exists a finite G-invariant regular Borel measure ρ on G/Γ. Then Γ has property T iff G has property (T).*

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[3]The original paper actually claims the result for real groups of rank$\geq$ 3 but it was pointed out later that the proof given there works over any local field and for rank 2 groups as well.