Math 100:V02 – SOLUTIONS TO WORKSHEET 14
RELATED RATES

1. Related Rates

(1) (Final 2018)

(a) Particle A travels with a constant speed of 2 units per minute on the x-axis starting at the point $(4, 0)$ and moving away from the origin, while particle B travels with a constant speed of 1 unit per minute on the y-axis starting at the point $(0, 8)$ and moving towards the origin. Find the rate of change of the distance between the two particles when the distance between the two particles is exactly 10 units.

Solution: At time t the first particle is at $(4 + 2t, 0)$ and the second is at $(0, 8 - t)$. The distance between them therefore satisfies

$$\begin{aligned} d(t)^2 &= (4 + 2t)^2 + (8 - t)^2 \\ &= 16 + 16t + 4t^2 + 64 - 16t + t^2 \\ &= 80 + 5t^2 \,. \end{aligned}$$

First, $d(t)^2 = 100$ when $t = 2$. Second at that time

$$2d\dot{d} = 10t = 20$$

so $\boxed{\dot{d}(2) = 1}$.

(b) Same question, but swap the velocities of the particles (particle A moves along the y axis, particle B moves along the x-axis).

Solution: At time t the first particle is at $(4, -t)$ and the second is at $(2t, 8)$ so the distance is now

$$\begin{aligned} d(t)^2 &= (2t - 4)^2 + (8 + t)^2 \\ &= 80 + 5t^2 \end{aligned}$$

and from this point the question is the same.

(2) A closed rectangular box has sides of lengths $4, 5, 6$cm. Suppose that the first and second sides are lengthening by $2\frac{\text{cm}}{\text{sec}}$ while the third side is shortening by $3\frac{\text{cm}}{\text{sec}}$.

(a) How fast is the volume changing?

Solution: Call the sides x, y, z. The volume is then $V(x, y, z) = xyz$. By the product rule

$$\frac{dV}{dt} = \dot{x}yz + x\dot{y}z + xy\dot{z}$$

so at the given time

$$\begin{aligned} \dot{V} &= 2 \cdot 5 \cdot 6 + 4 \cdot 2 \cdot 6 + 4 \cdot 5 \cdot (-3) \\ &= \boxed{48\frac{\text{cm}^3}{\text{sec}}\,.} \end{aligned}$$

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(b) How fast is the surface area changing?

Solution: The surface area is $A(x, y, z) = 2xy + 2yz + 2zx$. By the product rule

$$\frac{dV}{dt} = 2\left[\dot{x}(y + x) + \dot{y}(x + z) + \dot{z}(x + y)\right]$$

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so at the given time

$$\dot{A} = 2\left[2(5+6)+2(4+6)-3(4+5)\right]$$
$$= \boxed{30\frac{\text{cm}^2}{\text{sec}}}.$$

(c) How fast is the main diagonal changing?
Solution: The main diagonal's length satisfies $L^2 = x^2 + y^2 + z^2$. Differentiating we get

$$2L\dot{L} = 2x\dot{x} + 2y\dot{y} + 2z\dot{z}\,.$$

At the given time $L = \sqrt{4^2+5^2+6^2} = \sqrt{77}$ and hence

$$\dot{L} = \frac{1}{\sqrt{77}}\left[2\cdot 4 + 2\cdot 5 - 3\cdot 6\right]$$
$$= \boxed{0\frac{\text{cm}}{\text{sec}}}.$$

(3) Baseball is played on a square $HABC$ of side length 90ft. A player runs from corner A to B. How fast is the player running, if when she is half-way between corners A, B their distance to corner C is decreasing at the rate of $3\sqrt{5}\frac{\text{ft}}{\text{s}}$?
Solution: Let H be at the origin of the coordinates and let $A = (90,0)$, $B = (90,90)$, $C = (0,90)$. Let the runner be at position $(90, y(t))$ at time t. Then their distance from third base satisfies $d^2 = 90^2 + (y-90)^2$. Differentiating with respect to y we find

$$2\dot{d}d = 2(y-90)\dot{y}\,.$$

Now at the given time we have $y = 45$ and hence

$$d^2 = (2\cdot 45)^2 + (45)^2$$
$$= 45^2\left(2^2+1\right) = 45^2\cdot 5$$

so $d = 45\sqrt{5}$. We also have $\dot{d} = -3\sqrt{5}$ and therefore

$$\dot{y} = \frac{d\dot{d}}{y-90} = \frac{45\sqrt{5}\cdot(-3\sqrt{5})}{-45} = \boxed{15\frac{\text{ft}}{\text{s}}}.$$

(4) (CLP notes problem 3.2.2.14) The minute hand of a clock is 10cm long; the hour hand of the clock is 5cm long. How fast is the distance between the tips of the hands decreasing at 4 o'clock?
Solution: Let L, ℓ be the lengths of the hands. Let μ, η be their angles with the vertical (increasing clockwise, naturally). Then the hands make angle $\eta - \mu$ with each other, so by the law of cosines the distance between their tips is[1]

$$d^2 = L^2 + \ell^2 - 2L\ell\cos(\eta-\mu)\,.$$

At $4:00$ we have $\eta = \frac{4}{12}\cdot 2\pi = \frac{2}{3}\pi$ and $\mu = 0$ so

$$\cos(\eta-\mu) = \cos\left(\frac{2}{3}\pi\right)$$
$$= -\cos\left(\frac{1}{2}\pi + \frac{1}{6}\pi\right)$$
$$= -\sin\left(\frac{1}{6}\pi\right) = -\frac{1}{2}$$

and $d^2 = 10^2 + 5^2 - 100(-\frac{1}{2}) = 175$ so $d = 5\sqrt{7}$. Taking the derivative we get

$$2\dot{d}d = 2L\ell\sin\left(\eta-\mu\right)\cdot\left(\dot{\eta}-\dot{\mu}\right)$$

[1] This can also be computed by noting that the tips of the hand are at $L\left(\sin\mu, \cos\mu\right)$ and $\ell\left(\sin\eta, \cos\right)\eta$ by applying Pythagoras.

so

$$\dot{d} = \frac{L\ell}{d}\left(\dot{\eta} - \dot{\mu}\right)\sin\left(\eta - \mu\right) .$$

At the given time we have $\sin\left(\eta - \mu\right) = \sin\left(\frac{2}{3}\pi\right) = \sin\left(\frac{1}{3}\pi\right) = \frac{\sqrt{3}}{2}$. We have $\dot{\mu} = \frac{2\pi}{\mathrm{h}}$ and $\dot{\eta} = \frac{2\pi}{12\mathrm{h}}$ so

$$\begin{aligned}\dot{d} &= \frac{50}{5\sqrt{7}} \cdot \frac{2\pi}{\mathrm{h}}\left(\frac{1}{12} - 1\right) \cdot \frac{\sqrt{3}}{2} \\ &= -\frac{55\pi}{2\sqrt{21}}\end{aligned}$$

so the distance is decreasing at the rate of $\frac{55\pi}{2\sqrt{21}}\frac{\mathrm{cm}}{\mathrm{h}}$.

(5) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\mathrm{m}^3/\mathrm{min}$. How fast is the water rising when its height is 5m?

Solution: The water fills a conical volume inside the drain. Suppose that at time t the height of the water is $h(t)$ and the radius at the surface of the water is $r(t)$. Then by similar triangles

$$\frac{r(t)}{h(t)} = \frac{1}{6} .$$

We therefore have $r(t) = \frac{h(t)}{6}$. The volume of the water is therefore

$$V(t) = \frac{1}{3}\pi r^2 h = \frac{\pi}{108}h^3(t) .$$

Differentiating we find

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{36}h^2(t)\frac{\mathrm{d}h}{\mathrm{d}t} .$$

In particular, if $\frac{\mathrm{d}V}{\mathrm{d}t} = 5\mathrm{m}^3/\mathrm{min}$ and $h = 5\mathrm{m}$ then

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{36 \cdot 5}{\pi \cdot 5^2} = \frac{36}{5\pi}\frac{\mathrm{m}}{\mathrm{min}} .$$

(b) The drain is unclogged and water begins to drain at the rate of $(5 + \frac{\pi}{4})\mathrm{m}^3/\mathrm{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Solution: We are now given $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\pi}{4}\frac{\mathrm{m}^3}{\mathrm{min}}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = -1\frac{\mathrm{m}}{\mathrm{min}}$. Then

$$h(t) = \sqrt{\frac{36\frac{\mathrm{d}V}{\mathrm{d}t}}{\pi\frac{\mathrm{d}h}{\mathrm{d}t}}} = \sqrt{\frac{-36\pi}{4\pi(-1)}} = \sqrt{9} = 3\mathrm{m} .$$