# Math 100, Lecture 16, 7/8/2024

**Last time:** Autonomous ODE: $\dot{y} = F(y)$

(1) steady states/fixed points/equillibria are values $y_0$ s.t. $F(y_0) = 0$, so that $y(t) = y_0$ is a solution

(2) By determining sign of $F(y)$ between steady states can determine flow on phase line.

**Today:** Related Rates
(next time: optimization)

Situations: Problem specifies "setup". To solve it need to (a) define variables; (b) find function/relation connecting them; (c) do calculus (d) interpret result.

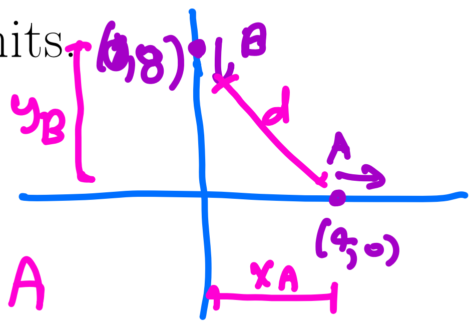
**Math 100 – WORKSHEET 11**
**RELATED RATES**

# 1. Related Rates

(1) (Final 2018)

(a) Particle A travels with a constant speed of 2 units per minute on the $x$-axis starting at the point $(4, 0)$ and moving away from the origin, while particle B travels with a constant speed of 1 unit per minute on the $y$-axis starting at the point $(0, 8)$ and moving towards the origin. Find the rate of change of the distance between the two particles when the distance between the two particles is exactly 10 units.

(0) picture →




(1) names: say particle A is at $(x_A(t), 0)$

B at $(0, y_B(t))$

distance between them is $d$

$t$ = time in minutes

$t \geq 0$

(2) relations:

$x_A(t) = 4 + 2t$

$y_B(t) = 8 - t$

$d^2 = x_A^2 + y_B^2$

$\Rightarrow d^2 = (4+2t)^2 + (8-t)^2$

$= 80 + 5t^2$

at the time $t$ where $d = 10$, have $80 + 5t^2 = 100$, so $t = 2$

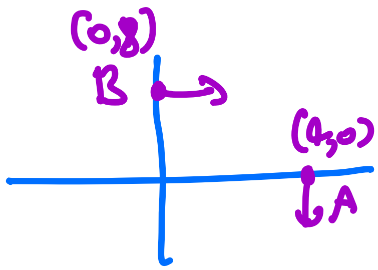
(3) calculus: $2d\dot{d} = 10t$

when $d = 10$, $t = 2$, $\boxed{\dot{d} = 1}$

(4) endgame: when $d = 10$, the distance is increasing at the rate of $\boxed{1 \frac{\text{unit}}{\text{min}}}$.

*Date*: 7/3/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) Same question, but swap the velocities of the particles (particle $A$ moves along the $y$ axis, particle $B$ moves along the $x$-axis).




location of A at time t: $(4, -t)$

-"- B -"- : $(2t, 8)$

$\Rightarrow d^2 = (2t-4)^2 + (8-(-t))^2$

$\vdots$

(2) A closed rectangular box has sides of lengths 4, 5, 6cm. Suppose that the first and second sides are lengthening by $2\frac{\text{cm}}{\text{sec}}$ while the third side is shortening by $3\frac{\text{cm}}{\text{sec}}$.

(a) How fast is the volume changing?

Call sides $x, y, z$. Then the volume is $V = xyz$.

then $\dot{V} = \dot{x}yz + x\dot{y}z + xy\dot{z}$ at given time $\dot{V} = 98 \frac{\text{cm}^3}{\text{sec}}$

(or: $V = (xy)\cdot z$ so $\frac{dV}{dt} = \frac{d(xy)}{dt}\cdot z + xy\cdot\frac{dz}{dt}$

$= \frac{dx}{dt}yz + x\frac{dy}{dt}z + xy\frac{dz}{dt}$)

$\underline{\text{or}}$: to 1st order, $x \cong 4+2t$, $y \cong 5+2t$, $z \cong 6-3t$

so $V \cong (4+2t)(5+2t)(6-3t) \cong 4\cdot5\cdot6 + (4\cdot5\cdot(-3) + 4\cdot6\cdot2 + 2\cdot5\cdot6)t$

$= 120 + 98t$

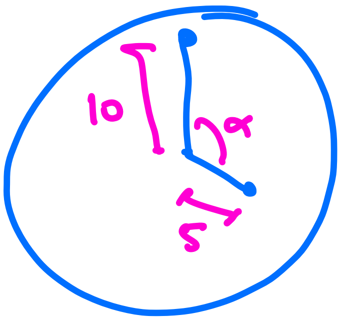
(b) How fast is the surface area changing?

$A = 2xy + 2xz + 2yz$ (6 faces, each rectangle bounded by two sides)

(c) How fast is the main diagonal changing?

$d^2 = x^2 + y^2 + z^2$

(4) (CLP notes problem 3.2.2.14) The minute hand of a clock is 10cm long; the hour hand of the clock is 5cm long. How fast is the distance between the tips of the hands decreasing at 4 o'clock?




$\theta$ = angle of minute hand

$\phi$ = " " hours "

$\alpha = \phi - \theta$

$d$ = distance

By law cosines: $d^2 = 10^2 + 5^2 - 2 \cdot 10 \cdot 5 \cdot \cos(\alpha)$

Or: tip of minute hand is at $10(\sin\theta, \cos\theta)$
hours " $5(\sin\phi, \cos\phi)$

so $2d\dot{d} = -100 \cdot -\sin(\alpha) \cdot \dot{\alpha}$

$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

At 4 o'clock $\alpha = \frac{1}{3} \cdot 2\pi$, $d^2 = 125 - 100\cos\left(\frac{2\pi}{3}\right)$
$= 175 = 25 \cdot 7$

$\alpha = \phi - \theta$ so $\dot{\alpha} = \dot{\phi} - \dot{\theta} = \frac{2\pi}{12}\frac{1}{\text{hour}} - \frac{2\pi}{\text{hour}} = -\frac{11}{12} \cdot \frac{2\pi}{\text{hr}}$
$= -\frac{11\pi}{6}\frac{1}{\text{hour}}$

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plug in

$2 \cdot 5 \cdot \sqrt{7} \cdot \dot{d} = 100 \cdot \sin\left(\frac{2\pi}{3}\right) \cdot \left(-\frac{11\pi}{6}\right) \cdot \frac{1}{\text{hour}}$

$\frac{\sqrt{3}}{2}$



so $\dot{d} = \frac{-100 \cdot \sqrt{3} \cdot 11\pi}{2 \cdot 5 \cdot \sqrt{7} \cdot 2 \cdot 6} \frac{1}{\text{hour}} = -\frac{55\pi\sqrt{3}}{6\sqrt{7}} \frac{\text{cm}}{\text{hr.}}$