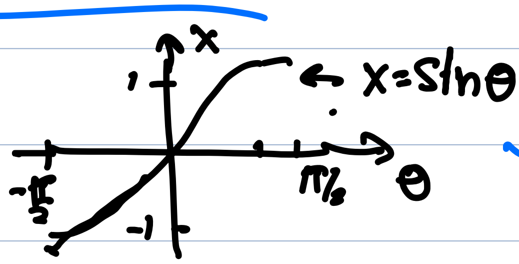
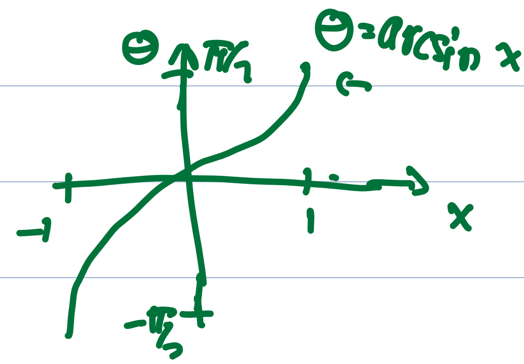
Math 100, lecture 13, 27/2/2024

Last time:





Today: (1) Logarithmic diff

(2) inverse trig.

Log. diff: if $y = f(x)$ then $\log y = \log f(x)$

so $\frac{y'}{y} = (\log f(x))'$

$$\Rightarrow \boxed{y' = f(x)\cdot(\log f(x))'}$$

$$\boxed{y' = y\cdot(\log y)'}$$

Math 100:V02 – WORKSHEET 11
INVERSE TRIG; LOGARITHMIC DIFFERENTIATION

1. Logarithmic differentiation

(1) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$

$\frac{d(\log(ax))}{dx} = a \cdot \frac{1}{ax} = \frac{1}{x}$

$\parallel$

$\frac{d(\log x + \log a)}{dx} = \frac{1}{x} + 0 = \frac{1}{x}$

$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) = \frac{1}{t^2+3t}(2t+3)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$

Date: 27/11/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) (Logarithmic differentiation) differentiate
$y = (x^2+1) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

$$\log y = \log(x^2+1) + \log \sin x + \left(-\tfrac{1}{2}\right)\log(x^3+3) + \cos x$$

$$\log(x^3+3)^{-\frac{1}{2}} = -\tfrac{1}{2}\log(x^3+3)$$

$$\frac{1}{y} y' = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$$

$$\Rightarrow y' = \underbrace{(x^2+1)\cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}}_{\text{mult by } y} \cdot \left[\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x\right]$$

(3) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) x^n if $y = x^n$ $\log y = \log(x^n) = n \cdot \log x$

$\Rightarrow$ (diff both sides) $\frac{1}{y} y' = n \cdot \frac{1}{x}$

$\Rightarrow y' = x^n \cdot n \cdot \frac{1}{x} = n \cdot x^{n-1}$ ✓

(b) x^x if $y = x^x$ $\log y = x \log x$

$$\Rightarrow \frac{1}{y} y' = \log x + x \cdot \frac{1}{x}$$

$$\Rightarrow y' = (\log x + 1) \cdot x^x$$

(c) $(\log x)^{\cos x}$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

take log

$\log y = \log(x^{\log x}) = (\log x)(\log x) = (\log x)^2$

diff both sides: $\frac{1}{y} y' = 2 \log x \cdot \frac{1}{x}$

solve for y: $y' = 2\log x \cdot \frac{1}{x} \cdot x^{\log x} = 2 \log x \cdot x^{\log x - 1}$.

(4) Let $f(x) = g(x)^{h(x)}$. Find a formula for f' in terms of g' and h'.

2. Inverse trig

(5) (evaluation)

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.

want θ s.t. $\sin\theta = -\frac{1}{2}$

Know: $\sin\frac{\pi}{6} = \frac{1}{2}$

So $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$\left(-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ so}\right)$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$$\boxed{\begin{gathered}\theta = \arcsin x \text{ iff} \\ (1)\ \sin\theta = x \\ (2)\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\end{gathered}}$$

need to solve $\sin\theta = \sin\frac{31\pi}{11}$

but: $\sin\frac{31\pi}{11} = \sin\left(\frac{31}{11}\pi - 2\pi\right)$

$= \sin\left(\frac{9}{11}\pi\right)$

$= \sin\left(\pi - \frac{9}{11}\pi\right) = \sin\left(\frac{2}{11}\pi\right)$

$\sin\theta = \sin(\pi - \theta)$

so $\arcsin\left(\sin\left(\frac{31}{11}\pi\right)\right) = \frac{2}{11}\pi$

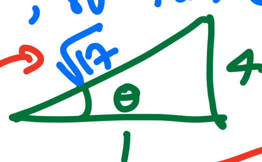
(b) (Final 2015) Simplify $\sin(\arctan 4)$

let $\theta = \arctan 4$, so $\tan\theta = 4$

draw triangle:

(fill lengths)

want $\sin\theta$



use Pythagoras to fill third side

$\sin\theta = \frac{4}{\sqrt{17}}$

(c) Find $\tan(\arccos(0.4))$

(6) Let $f(\theta) = \sin^2\theta + \cos^2\theta$. Find $\frac{df}{d\theta}$ without using trigonometric identities. Evaluate $f(0)$ and conclude that $\sin^2\theta + \cos^2\theta = 1$ for all θ.

(7) (Inverse functions)

(a) Suppose $g(x) = e^x$, $f(y) = \log y$. Show that $f(g(x)) = x$ and conclude that $(\log y)' = \frac{1}{y}$.

(b) Let $\theta = \arcsin x$. Find $\frac{d\theta}{dx}$. *Hint*: solve for x first.

$\boxed{\arcsin x \neq \frac{1}{\sin x}}$

if $\theta = \arcsin x$ then $\sin\theta = x$

so $\cos\theta \cdot \frac{d\theta}{dx} = 1$

so $\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}$

$\boxed{\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}}}$

need to find $\cos\theta$ in terms of x

so $\cos\theta = \sqrt{1-x^2}$

$$\frac{d(\arccos x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$$

(8) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$

(b) Find the line tangent to $y = \sqrt{1 + (\arctan(x))^2}$ at the point where $x = 1$.

(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?