Math 100, lecture 4, 18/1/2024

Last time: Asymptotics of expressions:

(1) if $f \ll g$ then $f+g \ll g$.

(2) if $f \sim \tilde{f}$, $g \sim \tilde{g}$ $\quad fg \sim \tilde{f}\tilde{g}$, $\frac{f}{g} \sim \frac{\tilde{f}}{\tilde{g}}$

$f \pm g \sim \tilde{f} \pm \tilde{g}$

$\Rightarrow$ use to compute limits (extract asymptotics, get limit)

Today: derivative.

Idea: get more precise asymptotics of f near $x=a$.

Fact: If f is defined by formula at & near a, have

$$\lim_{x \to a} f(x) = f(a)$$

If true (for any f), say f is continuous at a.

(if true for all a in some interval say f is continuous on the interval. Informally continuity means: "if x is close to a, $f(x)$ is close to $f(a)$".

Important case: "gluing functions":

If $f(x) = \begin{cases} g(x) & x > x_0 \\ h(x) & x < x_0 \end{cases}$

and g, h continuous through x_0 then f is continuous when $g(x_0) = h(x_0) = f(x_0)$

Example: Say $f(x) = \begin{cases} ax^2+b & x<0 \\ c\cdot\cos x & x>0 \end{cases}$

Clearly continuous at all $a \neq 0$ — there defined by formula)

cts at 0 iff $a\cdot 0^2+b = c\cdot\cos 0$
$\Leftrightarrow b = c = f(0)$

The Derivative

Suppose f is continuous at $x=a$, so if x close to a, $f(x)$ is close to $f(a)$. Q: "how close?"

To understand this, we will look at behaviour of the "distance" $f(x)-f(a)$. This is close to zero.
Questions what are the asymptotics of $f(x)-f(a)$ in terms of the small parameter $h = x-a \to 0$

(in terms of h, $x = a+h$, $f(x)-f(a) = f(a+h)-f(a)$)

Fact: For "most" functions in science: asymptotics is linear:

$$f(a+h) - f(a) \sim c \cdot h$$

then call c the derivative of f at a

write $f'(a) = c$, $\left.\frac{df}{dx}\right|_{x=a} = c$. Get:

$$f(a+h) - f(a) \sim f'(a)\, h$$

$$\Rightarrow$$

$$f(a+h) \approx f(a) + f'(a)\, h$$ ("linear approx")

Use in two ways: (1) Given f, by hand find linear approx, read off f'.

(2) If have "calculus" = symbolic rules for finding f', can compute linear approximation

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Or: $f(x) \approx f(a) + f'(a) \cdot (x-h)$

If $f(x)$ is close to $f(a) + f'(a) \cdot (x-h)$

then $\frac{f(x) - f(a)}{x-a} \approx f'(a)$, so $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$

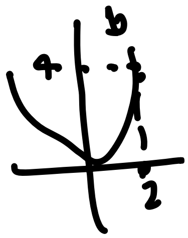
$\boxed{\text{"limit definition of derivative"}}$ $= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Math 100:V02 – WORKSHEET 3
THE DERIVATIVE

1. Three views of the derivative

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2, 4)$ is a point on the graph of $y = f(x)$.

$y = x^2$



(a) Let (x, x^2) be another point on the graph, close to $(2, 4)$. What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?

Slope: $\frac{\Delta y}{\Delta x} = \frac{x^2 - 4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \xrightarrow[x\to 2]{} 4$

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2+h)$ as $h \to 0$? What about $f(2+h) - f(2)$?

$f(2+h) = (2+h)^2 = 4 + 4h + h^2 \xrightarrow[h\to 0]{} 4 = f(2)$ ← continuity

$f(2+h) - f(2) = (4 + 4h + h^2) - 4 = 4h + h^2 \sim 4h \Rightarrow f'(2) = 4$

(c) What is $\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h}$?

$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} 4 + h = 4$

(b)
$f(2+h) \approx 4 + 4h$
$f(x) \approx 4 + 4(x-2)$

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2, 4)$?

Date: 18/1/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Line is $y = 4 + 4(x-2)$

Recap two points of view:

(1) from $f(x)=x^2$ figure out $f(2+h)=4+4h+h^2$
$\approx 4+4h$

$\Rightarrow$ set slope = derivative $=4$,
tangent line is $y=4+4(x-2)$

(2) from $f'(x)=2x$ set $f'(2)=4$ so
tangent line is $y=4+4(x-2)$

We found $f(2+h)\approx 4+4h$ $\boxed{h=x-2}$
$\Leftrightarrow$ $f(x)\approx 4+4(x-2)$ (working about $a=2$)

(2) ⋆⋆ An enzymatic reaction occurs at rate $k(T) = T(40 - T) + 10T$ where T is the temperature in degrees celsius. The current temperature of the solution is 20°C. Should we increase or decrease the temperature to increase the reaction rate?

$$k(20) = 20 \cdot (40 - 20) + 10 \cdot 20 = 600$$

$$k(20+h) = (20+h)(40-(20+h)) + 10(20+h)$$

$$= (20+h)(20-h) + 200 + 10h$$

$$= 600 + 10h - h^2 \approx 600 + 10h$$

to linear order in h

$\Rightarrow$ k increasing at 20, want to raise temperature

$\sin(h) = h - \frac{h^3}{6} + \frac{h^5}{120} - \frac{h^7}{5040} + \text{—} \ldots$

want slope of $\sin\theta$ at $\theta = 0$: 1

2. Definition of the derivative

Definition. $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a+h) \approx f(a) + f'(a)h$

(3) Find $f'(a)$ if

(a) $\star$ $f(x) = x^2$, $a = 3$.

(b) $\star\star$ $f(x) = \frac{1}{x}$, any a.

(c) $\star\star$ $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

(4) $\star\star$ Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h) - \cos 5}{h}$, $\lim_{x\to 0} \frac{\sin x}{x}$

(5) ⋆⋆⋆ (Final, 2015, variant – gluing derivatives) Is the function

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ x^2 \cos \frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?

3. The tangent line

(6) $\star$ (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

(7) $\star\star$(Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

(8) $\star\star\star$ Find the lines of slope 3 tangent to the curve $y = x^3 + 4x^2 - 8x + 3$.

(9) $\star\star\star$ The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

4. Linear approximation

Definition. $f(a+h) \approx f(a) + f'(a)h$

(10) Estimate

(a) $\star$ $\sqrt{1.2}$

(b) $\star$ (Final, 2015) $\sqrt{8}$

(c) $\star$ (Final, 2016) $(26)^{1/3}$