

Math 100A – WORKSHEET 8
TAYLOR EXPANSION

1. TAYLOR EXPANSION

- (1) (Review) Use linear approximations to estimate:
- (a) $\star \log \frac{4}{3}$ and $\log \frac{2}{3}$. Combine the two for an estimate of $\log 2$.

 - (b) $\star \sin 0.1$ and $\cos 0.1$.
- (2) Let $f(x) = e^x$
- (a) Find $f(0), f'(0), f^{(2)}(0), \dots$
 - (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.
 - (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.
 - (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.
 - (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \leq k \leq 3$.
- (3) Do the same with $f(x) = \log x$ about $x = 1$.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The n th order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x - a) + \cdots + c_n(x - a)^n$$

(4) ★ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

(5) ★★ Find the n th order expansion of $\cos x$, and approximate $\cos 0.1$ using a 3rd order expansion

(6) (Final, 2015) ★ Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f , expanded about $a = 3$. What is $f''(3)$?

(7) ★★ In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the “rest mass” of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

2. NEW EXPANSIONS FROM OLD

Near $u = 0$:	$\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \dots$	$\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \dots$
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- (8) ★ (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x + 1) \sin x$ about $x = 0$.

- (9) ★★ Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

- (10) ★★ Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

- (11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about $x = 0$.

- (12) (Change of variable/rebasing polynomials)

- (a) Find the Taylor expansion of the polynomial $x^3 - x$ about $a = 1$ using the identity $x = 1 + (x - 1)$.

(b) Expand e^{x^3-x} to third order about $a = 1$.

(13) Expand $\exp(\cos 2x)$ to sixth order about $x = 0$.

(14) *** Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x .

(15) (2023 Piazza @389) Find the asymptotics as $x \rightarrow \infty$

(a) ** $\sqrt{x^4 + 3x^3} - x^2$

(b) *** $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$

(16) Evaluate $\lim_{x \rightarrow 0} \frac{e^{-x^2/2} - \cos x}{x^4}$.