Math 100A – SOLUTIONS TO WORKSHEET 4
COMPUTING DERIVATIVES

1. Review of the derivative

(1) Expand $f(x+h)$ to linear order in h for the following functions and read the derivative off:

(a) $\star$ $f(x) = bx$

Solution: $b(x+h) - bx = bh$ so the derivative is $\boxed{b}$.

Solution: $b(x+h) = bx + bh$ so the derivative is $\boxed{b}$.

(b) $\star$ $g(x) = ax^2$

Solution: $a(x+h)^2 - ax^2 = 2axh + ah^2 \sim (2ax)h$ so the derivative is $\boxed{2ax}$.

Solution: $a(x+h)^2 = ax^2 + 2axh + ah^2 \approx ax^2 + (2ax)h$ so the derivative is $\boxed{2ax}$.

(c) $\star$ $h(x) = ax^2 + bx$.

Solution: $\left(a(x+h)^2 + b(x+h)\right) - \left(ax^2 + bx\right) = 2axh + ah^2 + bh \sim (2ax+b)h$ so the derivative is $\boxed{2ax+b}$.

Solution:

$$\begin{aligned} a(x+h)^2 + b(x+h) &= ax^2 + 2axh + ah^2 + bx + bh \\ &= \left(ax^2 + bx\right) + (2ax+b)\,h + ah^2 \\ &\approx \left(ax^2 + bx\right) + (2ax+b)\,h \end{aligned}$$

so the derivative is $\boxed{2ax+b}$.

Solution: $a(x+h)^2 \approx ax^2 + 2axh$ by part (a) and $b(x+h) = bx + bh$ by part (b) so

$$\begin{aligned} a(x+h)^2 + b(x+h) &\approx \left(ax^2 + 2axh\right) + (bx + bh) \\ &= \left(ax^2 + bx\right) + (2ax+b)\,h \end{aligned}$$

so the derivative is $\boxed{2ax+b}$.

(d) $\star\star$ $i(x) = \frac{1}{b+x}$

Solution: $\frac{1}{b+x+h} - \frac{1}{b+x} = \frac{(b+x)-(b+x+h)}{(b+x+h)(b+x)} = -\frac{h}{(b+x+h)(b+x)} \sim -\frac{h}{(b+x)^2}$ so the derivative is $\boxed{-\frac{1}{(b+x)^2}}$.

Solution:

$$\begin{aligned} \frac{1}{b+x+h} &= \frac{1}{b+x+h} - \frac{1}{b+x} + \frac{1}{b+x} \\ &= \frac{1}{b+x} + \frac{(b+x)-(b+x+h)}{(b+x+h)(b+x)} \\ &= \frac{1}{b+x} - \frac{h}{(b+x+h)(b+x)} \\ &\approx \frac{1}{b+x} - \frac{1}{(b+x)^2} \cdot h \end{aligned}$$

so the derivative is $\boxed{-\frac{1}{(b+x)^2}}$.

Date: 27/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(e) ⋆⋆⋆ $j(x) = 4x^4 + 5x$ (hint: use the known linear approximation to $2x^2$)
Solution: We have $j(x) = (2x^2)^2 + 5x$. Now $2(x+h)^2 \approx 2x^2 + 4xh$, so

$$\begin{aligned} f(x+h) &= \left(2(x+h)^2\right)^2 + 5(x+h) \\ &\approx \left(2x^2 + 4xh\right)^2 + 5(x+h) \\ &= 4x^4 + 16x^3h + 16x^2h^2 + 5x + 5h \\ &= (4x^4 + 5x) + (16x^3 + 5)h + O(h^2) \\ &\approx (4x^4 + 5x) + (16x^3 + 5)h \end{aligned}$$

so the derivative is $\boxed{16x^3 + 5}$.

2. Arithmetic of derivatives

(2) Differentiate
(a) ⋆ $f(x) = 6x^\pi + 2x^e - x^{7/2}$
Solution: This is a linear combination of power laws so $f'(x) = 6\pi x^{\pi-1} + 2ex^{e-1} - \frac{7}{2}x^{5/2}$.
(b) ⋆ (Final, 2016) $g(x) = x^2e^x$ (and then also $x^a e^x$)
Solution: Applying the product rule we get $\frac{dg}{dx} = \frac{d(x^2)}{dx} \cdot e^x + x^2 \cdot \frac{d(e^x)}{dx} = (2x + x^2)e^x = x(x+2)e^x$, and in general

$$\frac{d}{dx}(x^a e^x) = ax^{a-1}e^x + x^a e^x = x^{a-1}(x+a)e^x\,.$$

(c) ⋆ (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$
Solution: Applying the quotient rule the derivative is $\frac{2x\cdot(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$.
(d) ⋆ $\frac{x^2+A}{\sqrt{x}}$
Solution: We write the function as $x^{3/2} + Ax^{-1/2}$ so its derivative is $\frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$.

(3) ⋆ Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.
Solution: $f'(x) = \frac{1\cdot\left(\sqrt{x}+A\right) - x\left(\frac{1}{2}x^{-1/2}\right)}{\left(\sqrt{x}+A\right)^2} = \frac{\sqrt{x}+A-\frac{1}{2}\sqrt{x}}{\left(\sqrt{x}+A\right)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{\left(\sqrt{x}+A\right)^2}$. Plugging in $x = 4$ we have

$$\frac{3}{16} = f'(4) = \frac{1+A}{(2+A)^2}$$

so we have

$$3(2+A)^2 = 16(1+A)$$

that is

$$3A^2 + 12A + 12 = 16 + 16A$$

that is

$$3A^2 - 4A - 4 = 0\,.$$

In fact this gives $A = -\frac{2}{3}, 2$.

(4) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$.
(a) ⋆ What are the linear approximations to f and g at $x = 1$? Use them to find the linear approximation to fg at $x = 1$.
Solution: We have

$$\begin{aligned} f(x) &\approx f(1) + f'(1)(x-1) = 1 + 3(x-1) \\ g(x) &\approx g(1) + g'(1)(x-1) = 2 + 4(x-1) \end{aligned}$$

multiplying them we have

$$\begin{aligned}(fg)(x) &\approx (1+3(x-1))(2+4(x-1)) \\ &= 2 + 1\cdot 4(x-1) + 2\cdot 3(x-1) + 12(x-1)^2 \\ &\approx 2 + 10(x-1)\end{aligned}$$

to first order.

(b) $\star$ Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

Solution: $(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10.$

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{3\cdot 2 - 1\cdot 4}{2^2} = \frac{1}{2}\,.$$

(5) Evaluate

(a) $\star$ $(x\cdot x)'$ and $(x')\cdot(x')$. What did we learn?

Solution: $(x\cdot x)' = (x^2)' = 2x$ while $(x')\cdot(x') = 1\cdot 1 = 1$ – the "rule" $(fg)' = f'g'$ is **wrong**.

(b) $\star$ $\left(\frac{x}{x}\right)'$ and $\frac{(x')}{(x')}$. What did we learn?

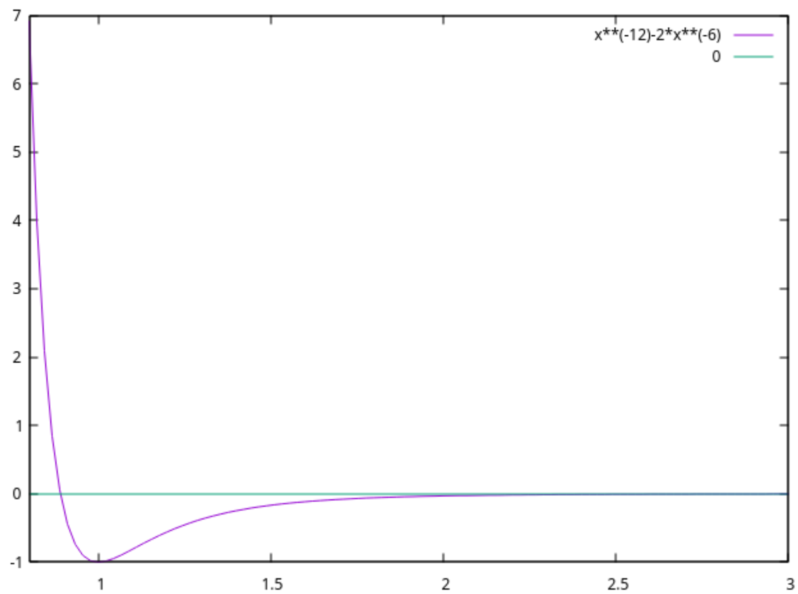
Solution: $\left(\frac{x}{x}\right)' = (1)' = 0$ while $\frac{(x')}{(x')} = \frac{1}{1} = 1$ – the "rule" $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$ is **wrong**.

(6) The *Lennart–Jones potential* $V(r) = \epsilon\left(\left(\frac{R}{r}\right)^{12} - 2\left(\frac{R}{r}\right)^{6}\right)$ models the electrostatic potential energy of a diatomic molecule. Here $r>0$ is the distance between the atoms and $\epsilon, R > 0$ are constants.

(a) $\star$ What are the asymptotics of $V(r)$ as $r\to 0$ and as $r\to\infty$?

Solution: For small r, $\frac{1}{r^{12}}$ blows up faster than $\frac{1}{r^6}$ so $V(r)\sim \epsilon\left(\frac{R}{r}\right)^{12}$ as $r\to 0$. For large r, $\frac{1}{r^{12}}$ decays faster than $\frac{1}{r^6}$ so $V(r)\sim -2\epsilon\left(\frac{R}{r}\right)^{6}$ as $r\to\infty$.

(b) Sketch a plot of $V(r)$.



Solution:

(c) Find the derivative $\frac{dV}{dr}(r) =$

Solution: $V(r) = \epsilon R^{12} r^{-12} - 2\epsilon R^6 r^{-6}$ so

$$\begin{aligned}V'(r) &= \epsilon R^{12}\cdot\left(-12r^{-13}\right) - 2\epsilon R^6\left(-6r^{-7}\right) \\ &= -12\epsilon R^{12} r^{-13} + 12\epsilon R^6 r^{-7} \\ &= 12\epsilon R^6 r^{-13}\left(r^6 - R^6\right)\,.\end{aligned}$$

(d) Where is $V(r)$ increasing? decreasing? Find its minimum location and value.

Solution: $V'(r)$ has the same sign as $r^6 - R^6$, so V' is negative when $r < R$ and is positive when $r > R$. We conclude that V is decreasing on $(0, R)$ and increasing on (R, ∞), and hence has a minimum at $r = R$, where $V(R) = \epsilon\,(1 - 2) = -\epsilon$. This makes ϵ the *binding energy* of the molecule.