Math 100C – SOLUTIONS TO WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. Asymptotics: simple expressions

(1) $\star$ Classify the following functions into *power laws / power functions* and *exponentials*: x^3, πx^{102}, e^{2x}, $c\sqrt{x}$, $-\frac{8}{x}$, 7^x, $8\cdot 2^x$, $-\frac{1}{\sqrt{3}}\cdot\frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, x^e, π^x, $\frac{A}{x^b}$.

Solution: Power laws: x^3, πx^{102}, $c\sqrt{x} = cx^{-1/2}$, $-\frac{8}{x} = -8x^{-1}$, $\frac{9}{x^{7/2}} = 9x^{-7/2}$, x^e, $\frac{A}{x^b} = Ax^{-b}$
Exponentials: $e^{2x} = (e^2)^x$, 7^x, $8\cdot 2^x$, $-\frac{1}{\sqrt{3}}\cdot\frac{1}{2^x} = -\frac{1}{\sqrt{3}}2^{-x}$, π^x.

(2) $\star$ How does the each expression behave when x is large? small? what is x is large but negative? $\star\star$ Sketch a plot

(a) $1 - x^2 + x^4$ ("Mexican hat potential")
Solution: When x is large, $7 + x^2 + x^4 \sim x^4$ (x^4 dominates both x^2 and the constant 7) while when x is small, $7 + x^2 + x^4 \sim 7$.

(b) $x^3 - x^5$
Solution: When x is very large, x^5 dominates x^3 so $x^3 - x^5 \sim -x^5$ (which is negative for x positive, positive for x negative!). When x is very small (close to zero), x^3 dominates (is bigger than x^5 though both are very small) and $x^3 - x^5 \sim x^3$.

(c) $e^x - x^4$
Solution: When x is very large, e^x dominates x^4 so $e^x - x^4 \sim e^x$. Near 0 we have $e^x \sim 1$ while x^4 is small, so $e^x - x^4 \sim 1$. As when x is large but negative e^x decays, so $e^x - x^4 \sim -x^4$.

(d) Wages in some country grow at 2% a year (so the wage of a typical worker has the form $A\cdot(1.02)^t$ where t is measured in years and A is the wage today). The cost of healthcare grows at 4% a year (so the healthcare costs of a typical worker have the form $B\cdot(1.04)^t$ where B is the cost today). Suppose that today's workers can afford their healthcare (A is much bigger than B). Will that be always true? Why or why not?
Solution: Asymptotically $(1.04)^t$ will dominate 1.02^t for large t, so eventually our assumptions must break down.

(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$A\cdot 1.05^t + B\cdot 1.1^t + C\cdot 0.98^t\,.$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?
Solution: When t is large, $(0.98)^t$ is actually decaying so this strain will disappear. On the other hand since $1.1 > 1.05$ over time 1.1^t will be much bigger than

(3) The (attractive) interaction between two hadrons (say protons) due to the strong nuclear force can be modeled by the *Yukawa potential* $V_{\mathrm{Y}}(r) = -g^2\frac{e^{-\alpha m r}}{r}$ where r is the separation between the particles, and g, α, m are positive constants. The elecctrical repulsion between two protons is described by the Columb potential $V_{\mathrm{C}}(r) = kq^2\frac{1}{r}$ where k, q are also positive constants. Which interaction will dominate for large distances? Will the net interaction be attractive or repulsive? Note that g^2 is much larger than kq^2.

Solution: At large distances the exponentially decaying factor will suppress the strong interaction, making the electrical interaction dominate. This is why nuclear fusion requires such high temperatures: we need to get the protons **really close** to each other for the strong force to take over, and this requires them moving very fast or the electrical repulsion will keep them apart.

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2. Asymptotics of complicated expressions

(4) Describe the following expressions in words

(a) $e^{|x-5|^3}$

Solution: This is the exponential, of the cube, of the absolute value, of $x-5$.

(b) $\frac{1+x}{1+2x-x^2}$

Solution: This is the ratio of (the sum of 1 and x) and (the sum of 1, $2x$, and $-x^2$).

(c) $\frac{e^x+A\sin x}{e^x-x^2}$

Solution: This is the ratio of (the sum of e^x and the product of A and $\sin x$) and (the difference of e^x and x^2).

(d) $\left(\frac{t+\pi}{t-\pi}\right)\sin\left(\frac{t+\pi}{2}\right)$

Solution: This is the product of (the ratio of the sum $t+\pi$ and the difference $t-\pi$) and the sign of the product of $\frac{1}{2}$ and the sum of t, π.

(5) For each of the functions in (a),(b),(c),(d) determine its asymptotics as $x \to 0$ and as $x \to \infty$.

(a) $\star$

Solution: (a) For x close to 0, $x-5 \sim -5$ so $|x-5| \sim 5$ so $|x-5|^3 \sim 125$ so $e^{|x-5|^3} \sim e^{125}$. For x very large $x-5 \sim x$ and since x is positive $|x-5| \sim |x| = x$ so $|x-5|^3 \sim x^3$. $e^{|x-5|^3}$ therefore grows roughly like e^{x^3} (in truth e^{x^3} is actually much bigger than $e^{(x-5)^3}$ – the ratio is on the scale of e^{15x^2}– but our expression captures the gist of the growth pattern).

(b) $\star$

Solution: (d) As $x \to 0$ x, x^2 are negligible next to the 1 so $\frac{1+x}{1+2x-x^2} \sim \frac{1}{1} = 1$. As $x \to \infty$ x dominates 1 so $x+1 \sim x$ and x^2 dominates $x, 1$ so $1+2x-x^2 \sim -x^2$. Thus $\frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} = -\frac{1}{x}$ – in other words the whole expression decays roughly like $\frac{1}{x}$.

(c) $\star\star\star\star$

Solution: (c) For x near 0 we have $e^x \sim e^0 = 1$ and $\sin x \to 0$ (we'll later learn that $\sin x \sim x$ near 0) so $e^x + A\sin x \sim 1$ near 0. Similarly $x^2 \sim 0$ so $e^x - x^2 \sim 1$ and we have $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{1}{1} = 1$. For large x we have $|\sin x| \leq 1$ so $A\sin x$ is much smaller than e^x and $e^x + A\sin x \sim e^x$. Simliarly e^x dominates any polynomial including x^2 and we have $e^x - x^2 \sim e^x$. Thus at infinity $\frac{e^x+A\sin x}{e^x-x^2} \sim \frac{e^x}{e^x} = 1$.

(d) $\star\star\star$

Solution: (d) As $t \to 0$ $\frac{t+\pi}{2} \sim \frac{\pi}{2}$ and the sin $\frac{\pi}{2} = 1$. Also π dominates t so $\frac{t+\pi}{t-\pi} \sim \frac{\pi}{-\pi} = -1$ thus $\left(\frac{t+\pi}{t-\pi}\right)\sin\left(\frac{t+\pi}{2}\right) \sim -1 \cdot 1 = -1$. As $t \to \infty$ t dominates π so $\frac{t+\pi}{t-\pi} \sim \frac{t}{t} = 1$ but $\sin\left(\frac{t+\pi}{2}\right)$ keeps oscillating, so there is no simple aysmptotic.