1A3

5. The Chain rule (4/10/2023)

Goals.

(1) The Chain Rule
(2) Logarithmic differentiation

Last Time. Arithmetic of derivatives

(1) **Linearity:** $(af+bg)'(x) = af'(x) + bg'(x)$ — a, b constants

(2) **Product rule:** $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

(3) **Quotient rule:** $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}$; $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Key tool: linear approximation

$$f(a+h) \approx f(a) + f'(a)h$$

Example: $\frac{d}{dx}(\sin(3x))$

Math 100A – WORKSHEET 5
THE CHAIN RULE

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.

(a) ⋆Expand $\sin(y+h)$ to linear order in h. Write down the linear approximation to $\sin y$ about $y = a$.

$\sin(y+h) \approx \sin(y) + (\cos y)h$

(b) ⋆⋆Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

$F(x+h) = \sin(3(x+h)) = \sin(3x+3h) \underset{y=3x}{=} \sin(y+3h)$

$\approx \sin y + (\cos y)\cdot 3h = \sin(3x) + \underbrace{\cos(3x)\cdot 3}_{\text{slope}}\cdot h$

So

$F'(x) = \frac{d}{dx}(\sin(3x)) = 3\cos(3x)$

Date: 4/10/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Write each function as a composition and differentiate

(a) $\star$ e^{3x} $\qquad g'(x) = 3 \qquad f'(y) = e^y$

$e^{3x} = f(g(x))$ where $g(x) = 3x$, $f(y) = e^y$

so $\frac{d}{dx}(e^{3x}) = f'(g(x)) \cdot g'(x) = e^{3x} \cdot 3$

or

let $y = 3x$. Then $\frac{d(e^{3x})}{dx} = \frac{d(e^y)}{dx} = \frac{d(e^y)}{dy} \cdot \frac{dy}{dx}$

$= e^y \cdot 3 = 3e^{3x}$

want answer in terms of x

(b) $\star$ $\sqrt{2x+1}$

$\frac{d}{dx}\left((2x+1)^{\frac{1}{2}}\right) = \frac{1}{2\sqrt{y}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

$f(y) = \sqrt{y}$, $g(x) = 2x+1$

(c) (Final, 2015) $\star$ $\sin(x^2)$ chain rule

let $\theta = x^2$. Then $\dfrac{d(\sin\theta)}{dx} = \dfrac{d(\sin\theta)}{d\theta} \cdot \dfrac{d\theta}{dx}$

$= \cos\theta \cdot 2x = \cos(x^2)\cdot 2x$

trig diff power law

(d) $\star$ $(7x + \cos x)^n$.

let $u = 7x + \cos x$ so $(7x+\cos x)^n = u^n$

then $\dfrac{d(u^n)}{dx} = \dfrac{d(u^n)}{du} \cdot \dfrac{du}{dx} = nu^{n-1}\cdot(7 - \sin x)$

$= n(7x+\cos x)^{n-1}(7-\sin x)$

or $g(x) = 7x + \cos x$, $f(y) = y^n$, $f'(y) = ny^{n-1}$, $g'(x) = 7 - \sin x$, ...

(3) (Final, 2012) $\star\star$ Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

By the chain rule, $f'(x) = g'(2\sin x)\cdot 2\cos x$

(Let $y = 2\sin x$, then $f(x) = g(y)$ so $\dfrac{df}{dx} = \dfrac{dg}{dx} = \dfrac{dg}{dy}\cdot\dfrac{dy}{dx}$

$= g'(y)\cdot 2\cos x$)

So $f'\left(\frac{\pi}{4}\right) = g'\left(2\sin\frac{\pi}{4}\right)\cdot 2\cos\left(\frac{\pi}{4}\right)$

$= g'(\sqrt{2})\cdot\sqrt{2} = \sqrt{2}\cdot\sqrt{2} = 2.$

(4) Differentiate

(a) ⋆ $7x + \cos(x^n)$

$$\frac{d}{dx}\left(7x + \cos(x^n)\right) = 7 - \sin(x^n) \cdot nx^{n-1}$$

(b) ⋆ $e^{\sqrt{\cos x}}$

$$\frac{d}{dx} e^{\sqrt{\cos x}} = e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}}(-\sin x) = -\frac{1}{2} e^{\sqrt{\cos x}} \cdot \frac{\sin x}{\sqrt{\cos x}}$$

$$\frac{d}{dx} e^{\sqrt{\cos x}} = \frac{d(e^{\sqrt{\cos x}})}{d(\sqrt{\cos x})} \cdot \frac{d(\sqrt{\cos x})}{d(\cos x)} \cdot \frac{d(\cos x)}{dx} = \ldots$$

(c) ⋆ (Final 2012) $e^{(\sin x)^2}$

$$\frac{d}{dx}\left(e^{(\sin x)^2}\right) = e^{(\sin x)^2} \cdot 2\sin x \cos x$$

(5) ⋆⋆ Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Differentiating both sides, we get by the chain rule

$$f'(g(x)) \, g'(x) = 3x^2 \quad \text{so} \quad f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$$

so $\boxed{g'(4) = \frac{48}{5}}$

Logarithms

By definition $\log x$ is the number y so that $x = e^y$
(i.e. we take logs to the **natural** base)

WS 6

Derivative of $y = \log x$.

Then $x = e^y$. Differentiating along the curve we get from chain rule

$$1 = \frac{d}{dx}(e^y) = e^y \cdot \frac{dy}{dx}$$

$$\left(\text{or: } (x)' = 1 = (e^y)' = e^y \cdot y'\right)$$

so $y' = \frac{dy}{dx} = \frac{d(\log x)}{dx} = \frac{1}{e^y} = \frac{1}{x}$

WS 17) etc

Aside: $\frac{d}{dx} \log|x| = \frac{1}{x}$ for all $x \neq 0$

Aside: $\log_b x = \frac{\log x}{\log b}$ so $\frac{d(\log_b x)}{dx} = \frac{1}{\log b} \cdot \frac{1}{x}$

2. Logarithmic differentiation

(6) ⋆ $\log\left(e^{10}\right) = 10$ $\qquad \log(2^{100}) = 100\log 2$

(7) ⋆ Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} = \frac{1}{ax}\cdot a = \frac{1}{x}$ $\qquad \frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2+3t\right) = \frac{2t+3}{t^2+3t}$

Or: $\log(ax) = \log a + \log x$

$h(r) = 2+\sin r$

$g(s) = \log s$

$f(t) = \frac{1}{t}$

$\searrow\ f(g(h(r)))$

(b) ⋆ $\frac{\mathrm{d}}{\mathrm{d}x}x^2\log(1+x^2) =$

$2x\log(1+x^2) + x^2\cdot\frac{2x}{1+x^2} = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$

or

pdt rule

$= 2x\cdot\log(1+x^2) + x^2\frac{d}{dx}\log(1+x^2)$

$= 2x\log(1+x^2) + x^2\cdot\frac{1}{1+x^2}\cdot 2x$

$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$

$-\frac{1}{(\log(2+\sin r))^2}\cdot\frac{1}{2+\sin r}\cdot\cos r$

$\frac{d\left(\frac{1}{\log(\,)}\right)}{\log(2+\sin r)}\cdot\frac{d(\log(2+\sin r))}{d(2+\sin r)}\cdot\frac{d(2+\sin r)}{dr}$

$=$

(8) ⋆⋆ (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

$\log y = \log(1+x^2) + \log(\sin x) - \frac{1}{2}\log(3+x^3) + \cos x$

differentiating.

$$\frac{1}{y}y' = \frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(3+x^3)} - \sin x$$

so

$$y' = \underbrace{(x^2+1)\sin x \frac{1}{\sqrt{x^3+3}}e^{\cos x}}\left(\frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(3+x^3)} - \sin x\right)$$

answer only in terms of x (if possible)

(9) Differentiate using $\boxed{f' = f \times (\log f)'}$
(a) ⋆ x^n

$$\frac{d}{dx}(x^n) = x^n \cdot \frac{d}{dx}(\log(x^n)) = x^n \frac{d}{dx}(n \log x) = x^n \cdot n \cdot \frac{1}{x}$$
$$= nx^{n-1}$$

(b) $\star$ x^x

$$(x^x)' = x^x \cdot (\log(x^x))' = x^x(x\log x)' = x^x\left(\log x + \frac{x}{x}\right)$$
$$= x^x(\log x + 1)$$

(c) $\star\star$ $(\log x)^{\cos x}$

$$((\log x)^{\cos x})' = (\log x)^{\cos x} \cdot (\cos x \cdot \log\log x)' =$$
$$= (\log x)^{\cos x}\left[-\sin x \log\log x + \cos x\,(\log\log x)'\right]$$
$$= (\log x)^{\cos x}\left[-\sin x \log\log x + \cos x \frac{1}{\log x} \cdot \frac{1}{x}\right]$$

(d) (Final, 2014) $\star$ Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x only.