2. Limits & Asymptotes (22/9/2023)

Goals.

(1) Limits of functions
(2) Existence and nonexistence of limits: blowup
(3) Asymptotes

Last Time. Studied **asymptotics** of expressions: idea of finding the "essential behaviour" of the expression in some limit (usually $x\to\infty$, $x\to-\infty$, or $x\to 0$)

Examples: $e^x - x^4 \sim e^x$ as $x\to\infty$

$\frac{1+7x}{3x^4}$ "blows up like x^{-4} at 0"

"asymptotic to $\frac{1}{3x^4}$"

Q WS 1

Q: What about $e^x - A\cos x$ near 0? If $A\neq 1$

$e^x - A\cos x \sim 1-A$

If $A=1$, $e^x - \cos x \sim x$ as $x\to 0$ (see next class)

Math 100C – WORKSHEET 2
LIMITS AND ASYMPTOTES

(1) Review of asymptotics: analyze the expression $\frac{e^x + A\sin x}{e^x - x^2}$ as $x \to \infty$, $x \to 0$, $x \to -\infty$.

As $x \to \infty$ $e^x + A\sin x \sim e^x$ ($A\sin x$ is always bounded)
$e^x - x^2 \sim e^x$ so $\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{e^x}{e^x} \sim 1$

As $x \to -\infty$, $e^x \to 0$, $A\sin x$ oscillates, $e^x - x^2 \sim -x^2$ $\Rightarrow$ no ~~clear~~ simple asymptotic but $\frac{e^x + A\sin x}{e^x - x^2} \to 0$ as $x \to -\infty$

As $x \to 0$ $e^x + A\sin x \sim 1$, $e^x - x^2 \sim 1$ so $\frac{e^x + A\sin x}{e^x - x^2} \sim 1$

1. Limits

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 5}\left(x^3 - x\right)$

The expression $x^3 - x$ is ~~defined~~ defined at $x=5$

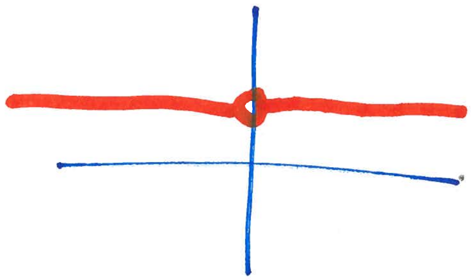
so $\lim_{x\to 5} x^3 - x = 5^3 - 5 = 120$

Date: 13/9/2023, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Today: Limits aka "what value would the function "like" to have at a point?"

Example: Let $f(x) = \frac{x}{x}$, defined for $x \neq 0$

Graph:



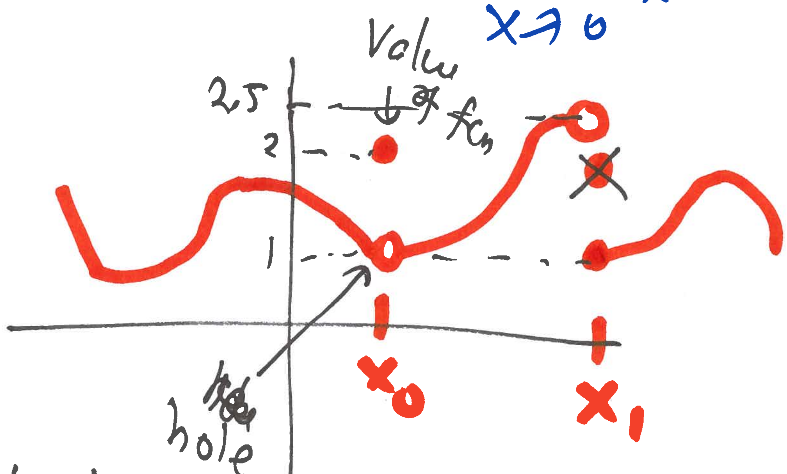
$f(x) = \begin{cases} 1 & x \neq 0 \\ \text{undef} & x = 0 \end{cases}$

not the same as

$g(x) = 1$ for all x

Even though $f(0)$ undefined, as $x \to 0$, $f(x)$ remains 1, we say: "the limit as $x \to 0$ of $f(x)$ is 1", write $\lim_{x \to 0} \frac{x}{x} = 1$

Example:



$f(x_0) = 2$

idea of limit: If we look at $f(x)$ for x near x_0 (but $x \neq x_0$) $f(x)$ will be close to 1, not 2!

$$\lim_{x \to x_0} f(x) = 1.$$

WS 2(a)

$\lim_{x \to x_0} f(x) = L$ means: "as x gets closer to x_0, $f(x)$ gets closer to L".

Easy case: f is defined at x_0, "well-behaved" there. Then $\lim_{x \to x_0} f(x) = f(x_0)$.

$\lim_{x \to 5} x^2 = 25$

Fact: If f is defined by formula, formula makes sense at x_0, then

$$\lim_{x \to x_0} f(x) = f(x_0)$$

In graphical example,

$\lim_{x \to x_1^-} f(x) = 2.5$ ← from left | $\lim_{x \to x_1^+} f(x) = 1$ ← from right

say $\lim_{x \to x_1} f(x)$ does not exist (DNE) piece-wise defined function

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

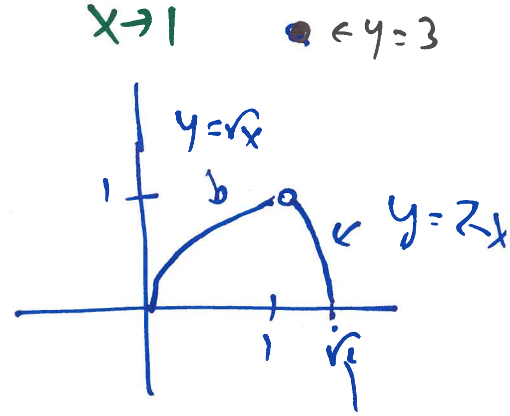
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$

if $x<1$, close to 1, $f(x) = \sqrt{x}$

agree

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x^2) = 2 - 1^2 = 1$

so $\lim_{x\to 1} f(x) = 1$



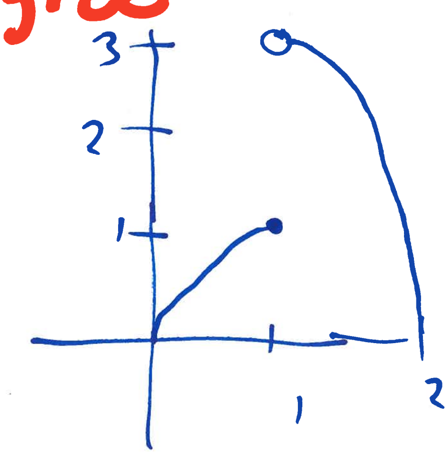
(c) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$

disagree

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4 - x^2 = 3$

so $\lim_{x\to 1} f(x)$ DNE.



Review

① **Asymptotics:** describing a function using a simpler function

($x^2 - e^x \sim -e^x$ as $x \to \infty$)

($\frac{1+e^x}{e^x} \sim 1$ as $x \to \infty$, 1 is the function 1)

② **Limits:** Finding what value a function "should" have at a point (if any)

[Both might not work out.]

③ Maybe have one sided limits.

④ functions can be defined piecewise

⑤ If lim DNE becaus f blows up at the point, extend the notion of limit, write $\lim f(x) = \infty$ or $= -\infty$ if true (need to test sign of $f(x)$ to determine this)

(3) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

[try $\frac{3-3}{3^2-3-12}$ get $\frac{0}{0}$] notice: $x^2+x-12=(x-3)(x+4)$

so $\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \lim_{x\to 3} \frac{x-3}{(x-3)(x+4)} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{7}$

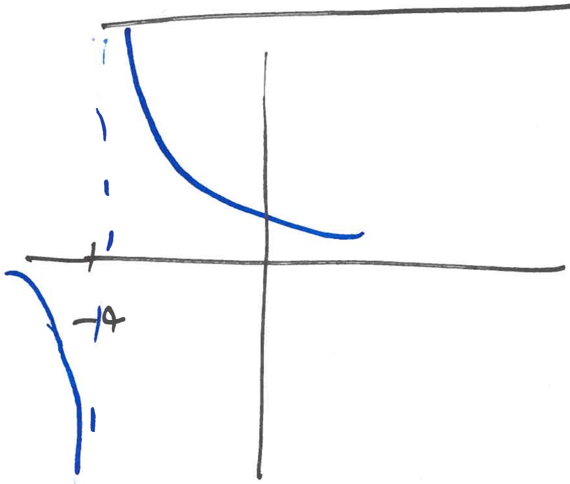
(b) What about $\lim_{x\to -4} f(x)$?

At $x=-4$, $f(x)$ undefined: $(-4)^2-4-12=0$

Near $x=-4$, $x\neq 3$, so $f(x)=\frac{1}{x+4}$

As $x\to -4$, $x+4\to 0$, so $\frac{1}{x+4}$ blows up

$\lim_{x\to -4} f(x)$ DNE



When $x>-4$, $x+4>0$, $\frac{1}{x+4}>0$,

$\lim_{x\to -4^+} f(x) = \infty$ (= DNE but escapes to ∞)

When $x<-4$, $x+4<0$, $\frac{1}{x+4}<0$, so

$\lim_{x\to -4^-} f(x) = -\infty$

(4) Evaluate

(a) $\lim_{x\to\infty} \frac{e^x + A\sin x}{e^x - x^2}$

(b) $\lim_{x\to 0} \frac{e^x + A\sin x}{e^x - x^2}$

(c) $\lim_{x\to -\infty} \frac{e^x + A\sin x}{e^x - x^2}$

(5) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1}$

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$.

here $x > -3$

As $x \to -3^+$, $x+2 \sim -1$ so $\frac{x+2}{x+3} \sim -\frac{1}{x+3} \xrightarrow[x\to -3^+]{} -\infty$

(c) $\lim_{x\to1}\frac{e^x(x-1)}{x^2+x-2}$ $= \lim_{x\to1}\frac{e^x(x-1)}{(x-1)(x+1)} = \lim_{x\to1}\frac{e^x}{x+2} = \frac{e}{3}$

Or: if $x\neq1$, $\frac{e^x(x-1)}{x^2+x-2} = \frac{e^x(x-1)}{(x-1)(x+2)} = \frac{e^x}{x+2} \xrightarrow[x\to1]{} \frac{e}{3}$

Wrong: $\lim_{x\to1}\frac{e^x(x-1)}{x^2+x-2} = \frac{e^x}{x+2} = e/3$

(d) $\lim_{x\to-2^-}\frac{e^x(x-1)}{x^2+x-2}$

(e) $\lim_{x\to1}\frac{1}{(x-1)^2}$

$\pi < 4 < 2\pi$ so $\sin 4 < 0$

(f) $\lim_{x\to4}\frac{\sin x}{|x-4|}$

As $x\to4$, $\frac{\sin x}{|x-4|} \sim \frac{\sin 4}{|x-4|} \xrightarrow[x\to4]{} -\infty$

(can see blowup)

(g) $\lim_{x\to\frac{\pi}{2}^+}\tan x$, $\lim_{x\to\frac{\pi}{2}^-}\tan x$.

If f blows up at x_0 (at least on one side) say it has the **vertical asymptote** $x=$

WS §(b), §(c), §(f)

If $\lim_{x \to x_0} f(x) = f(x_0)$ say f is "continuous" at x_0
(e.g. to glue pieces into cts fcn check values on both sides)

Can also compute limits as $x \to \infty$, $x \to -\infty$

If $\lim_{x \to \infty} f(x) = L$ (or at $-\infty$) say f has the **horizontal asymptote** $y = L$

Examples:

$$\lim_{x \to \infty} \frac{x^2}{x} = \lim_{x \to \infty} x = \infty$$

$$\lim_{x \to \infty} \frac{x}{x} = \lim_{x \to \infty} 1 = 1$$

$$\lim_{x \to \infty} \frac{x}{x^2} = \lim_{x \to \infty} \frac{1}{x} = 0$$

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x}$

as $x \to -\infty$, $\sqrt{4x^2+x} \sim \sqrt{4x^2} \sim -2x$ (or $|2x| = -2x$)

So $\frac{3x}{\sqrt{4x^2+x}-2x} \sim \frac{3x}{-2x-2x} \sim -\frac{3}{4}$

And $\lim_{x\to-\infty} (*) = -\frac{3}{4}$

Or: set $x = -y$, $y > 0$

compute $\lim_{y\to\infty} \frac{-3y}{\sqrt{4y^2-y}+2y}$