Math 535, lecture 35, 12/4/2023

Last time: Global Cartan involution

Thm: G s.s. (ctd) Lie alg $\mathfrak{g}$. Then have $\Theta \in \mathrm{Aut}(G)$ s.t.

(1) $\Theta^2 = \mathrm{Id}$, $\theta = d\Theta \in \mathrm{Aut}(\mathfrak{g})$ is a Cartan involution
($B_\theta(X,Y) = B(X, \theta Y) = \mathrm{Tr}(ad_X\, ad_{\theta Y} | \mathfrak{g})$ is neg. def.)

(2) $K = \mathrm{Fix}(\Theta) \subset G$ is closed, ctd, Lie $K = \mathrm{Fix}(\theta) = \mathfrak{k}$
$Z = Z(G) \subset K$ and K/Z is cpt.

(3) If $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\}$ then ("polar decomp")
the map $K \times \mathfrak{p} \to G$ is a diffeomorphism
$(k, X) \mapsto k \exp(X)$

Cor: K is a deformation retract of G
$\Rightarrow \pi_0(G) \simeq \pi_0(K)$, $\pi_1(G) \simeq \pi_1(K)$

Example: $G = SL_n(\mathbb{R})$, $\Theta(g) = {}^t g^{-1}$, $K = SO(n)$
$\mathfrak{g} = \mathfrak{sl}_n \mathbb{R}$ $\quad \theta(X) = -{}^t X$, $\mathfrak{k} = \{X = -{}^t X\}$
$\mathfrak{p} = \{X = {}^t X\}$

polar decomp: $g = QP$ where
Q is orthogonal, P is symm positive def.

Today: The Symmetric space

Recall: A Riemannian manifold is a pair (M,g) when M is a manifold (ctd) and g is a metric on M, i.e. a symm pos. def. section of bundle $\mathrm{Hom}(TM, T^*M)$.

I.e. at every $x \in M$, g_x is a pos. def. quadratic form on T_xM.

Fix (M,g). If $\gamma : [a,b] \to M$ is C^1 define

$$L(\gamma) = \int_a^b |d\gamma(t)|_{g_{\gamma(t)}} \, dt$$

(indep of parametrization of γ)

Define: $d_g(x,y) = \inf \left\{ L(\gamma) \;\middle|\; \begin{array}{l} \gamma : [a,b] \to M \\ \gamma(a) = x,\ \gamma(b) = y \end{array} \right\}$

Clearly a metric, in fact a length metric. Locally bilipschitz to Euclidean metric, so defines same topology, minimisers exist.

Minimisers (geodesics) exist, satisfy ODE
$\Rightarrow$ smooth.

Call any solution to ODE a geodesic.
(= locally length-minimizing curve)

Riemannian exponential map is the solution map with initial condition (p, X) : $p \in M$, $X \in T_pM$

Fact: the metric d_g is complete iff $\exp_p$ is defined on all of T_pM for all p.

$\exp_p$ is a local diffeo $\Rightarrow$ can reverse geodesics locally (exchange $\exp_p(X) \leftrightarrow \exp_p(-X)$)

Def: A Riemannian manifold (M,g) is a symmetric space if geodesic reversal extends to a global isometry of (M,g).

Examples: S^n is a symmetric space.
(if $S^n = \{(x_0, .., x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + ... + x_n^2 = 1\}$
map $(x_0, \underline{x}) \to (x_0, -\underline{x})$ which reverses about $(1, \underline{0})$)

Example: Positive-definite matrices

(1) inner prod = map $g: V \to V^*$ s.t. $g^* = g$
& g is pos-def.

$\mathcal{P}(V)$ = pos. def. symmetric elements of $Hom(V, V^*)$
tangent space = symm matrices

$$G_g(X,Y) = Tr(g^{-1} X g^{-1} Y)$$

(2) If $g, h \in \mathcal{P}(V)$ diagonalise g wrt h
with e.v. $\lambda_1, \ldots, \lambda_n$ then $d(g,h) = \left(\sum_i (\log \lambda_i)^2\right)^{1/2}$.

In POV (1), $u \in GL(V)$ acts on $\mathcal{P}(V)$
by $g \mapsto u^* g u$.
an isometry, action transitive, stabiliser of id
$O(n)$.

Fix a symmetric space S.
Lemma: $G = Isom(S)$ is transitive.
Pf: reverse at middle of geodesic between x, y

Fact: $Isom(M,g)$ is a Lie gp.

Set $K = Stab_{Isom(S)}(x)$, then K acts on T_xS

map $K \to O(T_xS, g_x)$ is injective, closed

$\Rightarrow K$ is compact.

Fact: Call a symmetric space S reducible if $S = S_1 \times S_2$ for symmetric spaces S_1, S_2

Then every symm space is a pdt of irreducible symm spaces. Those are either:

(1) $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$ ("flat")

(2) compact (e.g. $S^n = O(n)/O(n-1)$)

(3) noncompact.

Thm: If S is a symmetric space with only noncompact factors then $G = Isom(S)$ is s.s.

K = fixed points of Cartan involution, $S = G/K$
the involution is geodesic reversal

On G/K have natural metric: restrict Killing form to $\mathfrak{p} = T_e(G/K)$
(K-inv't on $\mathfrak{p}$ so G-inv't on G/K)

<u>Thm:</u> This is a symmetric space if G is s.s.
(easy: Cartan involution is -1 on $\mathfrak{p}$)

Polar decomp = Riemannian exponential map

<u>Fact:</u> If $S = G/K$ is non-compact, it has nonpositive sectional curvature

Also s.c. ($G/K \cong \mathfrak{p}$)

<u>Fact:</u> Let H be a cpt gp acting by isometries on a s.c. manifold of curvature ≤ 0. Then H fixes a point.

<u>Cor:</u> Any cpt subgp of G is conj. to a subgp of K.

Pf: If $H < G$ is cpt, H fixes a coset $gK \in S = G/K$

so $g^{-1}Hg \subset K$. $\Rightarrow$ K is a max'l cpt subgp

(Aside: saw if $\tilde{G}$ covers G, $\tilde{K}$ covers K,
$\tilde{G}/\tilde{K} = G/K$)

Example: $\mathbb{H} = \{ \overset{z}{x + iy} \mid \begin{matrix} x \in \mathbb{R} \\ y \in \mathbb{R}_{>0} \end{matrix} \}$ metric $\frac{dx^2 + dy^2}{y^2}$

If $x \in \mathbb{R}$ $n(x) \cdot z = z + x$ is an isometry

If $t \in \mathbb{R}$, $a(t) \cdot z = e^t z$ " " "

Can write these as $z \mapsto \frac{1 \cdot z + x}{0 \cdot z + 1}$

$$z \mapsto \frac{\sqrt{a} \cdot z + 0}{0 \cdot z + 1/\sqrt{a}}.$$

action of $N = \{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \}$, $A = \{ \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} \}$
via fractional linear transformations.

Transitive: $n(x) \cdot a(t) \cdot i = x + e^t i$.
(simply)

$\Rightarrow$ If $G = \text{Isom}(\mathbb{H})$ $\quad G = NAK$
where $K = \text{Stab}_G(i)$

$\exists$ Möbius transf. $f : (\mathbb{H}, i) \to (\mathbb{D}, 0)$
check: metric on $\mathbb{D}$ is invariant under rotation

$\Rightarrow$ $K = O(2)$

$\Rightarrow$ $G = PGL_2(\mathbb{R})$, $\mathbb{H}$ symmetric space

$$\begin{array}{c} G = PGL_2(\mathbb{R}) \\ \cup \\ SL_2(\mathbb{R}) \to PSL_2(\mathbb{R}) = G^0 \end{array}$$