Math 535, lecture 29 22/3/2023

Last time: $\mathfrak{g} = \mathrm{Lie}\, G$, G cpt ctd, . . ., $\mathfrak{n} = \bigoplus_{\beta>0} \mathfrak{g}_\beta$

Thm: let V be a f.d. irrep of G with highest wght λ.
Then:

(1) $\dim_{\mathbb{C}} V_\lambda = 1$

(2) $V_\lambda = \{ \underline{v} \in V \mid \mathfrak{n} \cdot \underline{v} = 0 \}$

(3) $V_\mu \neq 0 \Rightarrow \mu = \lambda - \sum_{i=1}^{r} n_i \alpha_i$, $n_i \in \mathbb{Z}_{\geq 0}$; $\Delta = \{\alpha_i\}_{i=1}^{r}$

(4) $\dim_{\mathbb{C}} V_{w\mu} = \dim_{\mathbb{C}} V_\mu$ $(w \in W)$

also

$V_\mu \neq 0 \Rightarrow |\mu| \leq |\lambda|$, equality iff $\mu \in W \cdot \lambda$.

($\Rightarrow \mu \in \mathrm{Conv}(W \cdot \lambda)$)

(5) V is determined by λ up to isom.

(from theory for G_α, λ is alg. integral: $\lambda(\check{\alpha}) \in \mathbb{Z}$)
$\Rightarrow$ all weights are

From pf: If V is an $\mathfrak{g}$-module, $\underline{v} \in V$ has weight μ, annihilated by $\mathfrak{n}$ ($\Leftrightarrow X_\alpha \underline{v} = 0$ for $\alpha \in \Delta$). Then submodule generated by $\underline{v}$ is

$$\mathcal{U}(\bar{\mathfrak{n}}) \cdot \underline{v}$$

with weights $\mu - \sum_{i=1}^{r} n_i \alpha_i$, $(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \cdot \underline{v})_\mu = \mathbb{C}\underline{v}$.

Today: converse: for each alg. integral dominant weight λ $\exists$ irrep of $\mathfrak{g}$ with highest weight λ.

Def: Call a rep'n (possibly ∞-dim) of $\mathfrak{g}$ (so of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$) a highest weight module if it is generated (as a rep'n) by a vector $\underline{v} \in V$ of weight λ s.t. λ is the highest occuring weight in V.

Again, if V is a highest weight module, $\underline{v}_\lambda \in V_\lambda$.
Then $\mathfrak{n} \cdot \underline{v}_\lambda = \underline{0}$ (λ is highest)
So $V = \mathcal{U}(\bar{\mathfrak{n}}) \cdot \underline{v}_\lambda$.

$\Rightarrow$ V is a sum of weight spaces

$\Rightarrow$ weights are of form $\lambda - \sum_{i=1}^{r} n_i \alpha_i \quad n_i \in \mathbb{Z}_{\geq 0}$

$\dim V_\lambda = 1$

Def: For a weight λ let $\mathbb{C}^\lambda$ for the 1d rep'n of $\mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$ where $\mathfrak{t}_{\mathbb{C}}$ acts via λ, $\mathfrak{n}$ acts by zero. The Verma module W^λ is the induced module

$$\mathrm{Ind}_{\mathcal{U}(\mathfrak{t}_{\mathbb{C}})\mathcal{U}(\mathfrak{n})}^{\mathcal{U}(\mathfrak{g}_{\mathbb{C}})} \mathbb{C}^\lambda \cong \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n})} \mathbb{C}^\lambda \cong \mathcal{U}(\mathfrak{g}_{\mathbb{C}})/I_\lambda$$

where I_λ is the left ideal of $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ generated by $\mathfrak{n} \cup \{H - \lambda(H)\}_{H \in \mathfrak{t}}$.

<u>Observation</u>: (1) This is a highest weight module.
(2) It is universal among them
(If V highest-weight module, map $\mathbb{C}^\lambda \ni 1 \to v_\lambda \in V_\lambda$ then map $X \cdot 1 \to X v_\lambda$ for $X \in \mathcal{U}(\mathfrak{g}_0)$)
well-defined by universal property of $\otimes$.

(3) Each weight space is fin-d.
(4) If $U \subset W^\lambda$ is a submodule, U is also a sum of weight spaces

$\Rightarrow$ (5) Every proper invariant subspace is contained in $\bigoplus_{\mu < \lambda} W^\lambda_\mu$. (a sum of weight spaces, can't meet W^λ_λ)

$\Rightarrow$ sum of all proper submodules is a proper submodule, hence the maximal one.

(6) W^λ has a unique irred quotient L^λ, the unique irreducible highest-weight module.

Thm: (Existence) Suppose λ is algebraically integral and dominant. Then L^λ is f.d.

Theme: To classify unitary dual (= f.d. irreps of G) First classify more general class, then see which are unitarizable)

Pf: let $v_\lambda \in W^\lambda_\lambda$ be the highest weight vector. Normalise X_α, $X_{-\alpha} = \overline{X_\alpha}$ s.t. $[X_\alpha, X_{-\alpha}] = \check{\alpha}$, then

$$X_\alpha (X_{-\alpha})^{k+1} = (X_{-\alpha})^{k+1} X_\alpha + (k+1) X_{-\alpha}^{k} (\check{\alpha} - k)$$

(in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$)

Apply to v_λ. $X_\alpha \cdot v_\lambda = 0$ (λ highest weight)

$\check{\alpha} \cdot v_\lambda = \lambda(\check{\alpha}) \cdot v_\lambda$

$$\Rightarrow X_\alpha \cdot (X_{-\alpha})^{k+1} v_\lambda = (k+1)(\lambda(\check{\alpha}) - k) \cdot (X_{-\alpha})^k v_\lambda$$

choosing $k = \lambda(\check{\alpha}) \in \mathbb{Z}_{\geq 0}$ by hypothesis we get

$$X_\alpha \cdot \left((X_{-\alpha})^{\lambda(\check{\alpha})+1} v_\lambda \right) = 0.$$

If β is another simple root, $X_{-\alpha}, X_\beta$ commute since $\mathfrak{g}_{\beta-\alpha} = \{0\}$. So

$$X_\beta\left((X_{-\alpha})^{\lambda(\alpha^\vee)+1} v_\lambda\right) = (X_{-\alpha})^{\lambda(\alpha^\vee)+1} \cdot X_\beta v_\lambda = \underline{0}.$$

$\Rightarrow \mathcal{U}(\mathfrak{g}_\mathbb{C}) \cdot \left((X_{-\alpha})^{\lambda(\alpha^\vee)+1} v_\lambda\right)$ is a highest-weight module of weight $\lambda - (\lambda(\alpha^\vee)+1)\alpha < \lambda$

$\Rightarrow$ in L^λ, the $\mathrm{Lie}\, G_\alpha$-submodule generated by v_λ is of dim $\lambda(\alpha^\vee)+1$

Write M_α = sum of all f.d. $\mathrm{Lie}\, G_\alpha$-submodules of L^λ.

Then $\mathfrak{g}_\mathbb{C} \otimes_\mathbb{C} M_\alpha$ is also a sum of f.d. $\mathrm{Lie}\, G_\alpha$-submodules

So its image in L^λ is such a module, hence contained in M_α

i.e.

$$\mathfrak{g}_\mathbb{C} \cdot M_\alpha \subset M_\alpha$$

So $M_\alpha \subset L^\lambda$ is a $\mathcal{U}(\mathfrak{g}_\mathbb{C})$-submodule

But $M_\alpha \neq \{0\}$ so $M_\alpha = L^\lambda$ (irreducibility).

Since f.d. rep'ns of Lie G_α are completely reducible and weights are S_α-inv't get

$$\dim_{\mathbb{C}} L^\lambda_{s_\alpha \mu} = \dim_{\mathbb{C}} L^\lambda_\mu$$

for all weights μ.

Now s_α generate W, so same holds for w.

Since weights spaces in W^λ are f.d. enough to show L^λ has finitely many weights

By W-invariance and finiteness of W enough to show finitely many W-orbits, ie finitely many <u>dominant</u> weights

If $\lambda - \sum_{i=1}^n n_i \alpha_i$ is dominant then

$$0 \leq \langle \rho, \lambda - \sum_{i=1}^n n_i \alpha_i \rangle \underset{\rho}{\Rightarrow} 0 \leq n_i \leq \frac{\langle \rho, \lambda \rangle}{\langle \rho, \alpha_i \rangle}.$$

$\langle \rho, \alpha_i \rangle, \langle \rho, \lambda \rangle \geq 0$

$\blacksquare$