Math 535, Lecture 27 17/3/2023

Setup: G ctd cpt Lie gp, max'l torus T

$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\beta \in \Phi} \mathfrak{g}_\beta = \bar{\mathfrak{n}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$, $\quad \mathfrak{n} = \bigoplus_{\beta > 0} \mathfrak{g}_\beta$

$\Phi = \Phi(G:T)$ roots, $W = W(G:T)$ Weyl gp $\quad \bar{\mathfrak{n}} = \bigoplus_{\beta < 0} \mathfrak{g}_\beta$

$\mathfrak{u}_\alpha = \ker \alpha \subset \mathfrak{t}$, $G_\alpha = Z_G(\mathfrak{u}_\alpha)$; $\mathrm{Lie}(G_\alpha) = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ (ss rk 1)

Fixed Weyl chamber C, walls $\{\mathfrak{u}_\alpha\}_{\alpha \in \Delta}$: Δ "simple roots"
Each $\beta \in \Phi^+$ has form $\sum_{\alpha \in \Delta} n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}_{\geq 0}$.

$\mathrm{Span}_{\mathbb{R}} \Delta = \mathfrak{t}/\mathfrak{z}$ where $\mathfrak{z} = Z(\mathfrak{g}) = \mathrm{Lie}\, Z(G)$

Defined coroots $\underset{\cap\, \mathfrak{t}}{\check{\alpha}}$ s.t. action of s_α on $\mathfrak{t}^*$ is via

$$s_\alpha(\nu) = \nu - \nu(\check{\alpha})\alpha$$

$\Rightarrow$ identify $\mathfrak{t}, \mathfrak{t}^*$ via invt inner pdt $\check{\alpha} = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$

$\Rightarrow \check{\alpha} \perp \mathfrak{z}$, $\mathrm{Span}_{\mathbb{R}} \{\check{\alpha}\}_{\alpha \in \Delta} = \mathfrak{z}^\perp$.

Last time: From $\mathfrak{g}$ as vsp created tensor algebra $T(\mathfrak{g})$ from Lie alg. structure the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Saw: Lie alg. hom $\mathfrak{g} \to A$ (A assoc alg) in bijection with alg. hom.s $\mathcal{U}(\mathfrak{g}) \to A$.

If $\{X_i\}_{i \in I}$ ordered basis of $\mathfrak{g}$ then ordered monomials $\{X_\sigma \mid \underset{\text{non decr}}{\sigma : [n] \to I}\}$ span $\mathcal{U}(\mathfrak{g})$

Thm: (Poincaré-Birkhoff-Witt) These are a basis of $\mathcal{U}(\mathfrak{g})$.

Cor: If $\mathfrak{g}_{\mathbb{C}} = \bar{\mathfrak{n}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$ then

$$\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\bar{\mathfrak{n}})\,\mathcal{U}(\mathfrak{t}_{\mathbb{C}})\,\mathcal{U}(\mathfrak{n})$$

Study irred f.d. rep (π, V) of $\mathfrak{g}$ (equiv of $\mathfrak{g}_{\mathbb{C}}$, equiv of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$).

Goal: Classify these, determine which are rep'ns of G.

Step 1: uniqueness

Ex: The lexicographic order makes $\mathbb{R}^r$ an ordered group

Enumerate $\Delta = \{\alpha_i\}_{i=1}^r$, $r =$ s.s. rk

Def: Call $\nu \in \mathfrak{t}^*$ positive if the first nonzero entry of $(\nu(\alpha_i^\vee))_{i=1}^r$ is positive. Write $\nu > \nu'$ if $\nu - \nu' > 0$.

Remark: insensitive to $\nu|_{\mathfrak{z}}$.

Lemma: let ν, ν', ν'' be distinct weights of V. Then (1) if $\nu > \nu'$, $\nu' > \nu''$ then $\nu > \nu''$.

(2) $\nu|_{\mathfrak{z}} = \nu'|_{\mathfrak{z}}$

(3) either $\nu > \nu'$ or $\nu' > \nu$.

Pf: (1) is the fact that $\mathbb{R}^r$ is an ordered gp.

$(0, 0, 0, , 0, +, ?, ?,)$

$- (0, \cdot, 0, +, ?, ?, ?, ?)$

(2) Schur's lemma.

(3) $\nu-\nu'$ vanishes on $\mathfrak{z}$, but is nonzero so $(\nu-\nu')(\check{\alpha}_i) \neq 0$ for some i.

<u>Cor:</u> Action of $\mathfrak{t}$ on V has a unique highest weight λ.

<u>Lemma:</u> For all $H \in \mathfrak{t}_{\mathbb{C}}$, $\pi(H)$ is diagonalizable $\Leftrightarrow$ V is a sum of weight spaces.

<u>Pf:</u> $\mathfrak{z}$ acts by a character by Schur's lemma, since $\mathfrak{z} \cup \{\check{\alpha}\}_{\alpha \in \Delta}$ span $\mathfrak{t}$, which is commutative, enough to show $\pi(\check{\alpha})$ are diagonalizable.

But that $\check{\alpha}$ acts diagonally follows by restricting rep'n to G_α, or Lie G_α, or to $\underset{\mathfrak{sl}_2\mathbb{C}}{\underset{\cong}{\mathrm{Span}\{X_\alpha, \bar{X}_\alpha, \check{\alpha}\}}}$

(since $[X_\alpha, \bar{X}_\alpha]$ prop. to $\check{\alpha}$). ∎

<u>Lemma:</u> The highest weight λ of V is <u>dominant</u> (in positive dual chamber) and algebraically integral

Pf: Restrict to Lie G_α for $\alpha \in \Delta$. Then λ still highest weight of V, so $\lambda(\check{\alpha}) \geq 0$ (the highest weight is $2l$ if dim is $2l+1$). and $\lambda(\check{\alpha}) \in \mathbb{Z}$.

Thm: (highest weight) V f.d. irrep of $\mathfrak{g}$, λ the highest weight. Then: I

(1) $\dim_{\mathbb{C}} V_\lambda = 1$.

$\mathfrak{n} = \bigoplus_{\beta > 0} \mathfrak{g}_\beta$

(2) $V_\lambda = \{ \underline{v} \in V \mid \mathfrak{n} \cdot \underline{v} = \underline{0} \}$

(3) Every weight of V has form $\lambda - \sum_{i=1}^{r} n_i \alpha_i$, with $n_i \in \mathbb{Z}_{\geq 0}$

(4) For $w \in W$, $\mu \in \mathfrak{t}^*$, $\dim_{\mathbb{C}} V_{w\mu} = \dim_{\mathbb{C}} V_\mu$. moreover all weights satisfy $|\mu| \leq |\lambda|$, with equality iff $\mu \in W \cdot \lambda$.

(5) V is uniquely determined by λ.

Later: Thm: II If λ is dominant, a.g. integral, there exists an irrep of $\mathfrak{g}$ with highest weight λ.

Thm III: The rep'n integrates to G iff $\lambda \in \Lambda^*$.