Math 535, lecture 26 15/3/2023

Last time: G of s.s. rk 1, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathbb{C}e \oplus \mathbb{C}f$

$h = [e,f] \in \mathfrak{t}_{\mathbb{C}}$ s.t. $[h,e] = 2e$, $[h,f] = -2f$.

$\mathfrak{t}_{\mathbb{C}} = \mathbb{C}h \oplus \mathfrak{z}_{\mathbb{C}}$ $(\mathfrak{z} = Z(\mathfrak{g}))$

① irreps of $\mathfrak{g}_{\mathbb{C}}$ have the form $\mathrm{Span}\{v_m\}_{m=-\ell}^{\ell}$
where $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$;

$\mathfrak{z}_{\mathbb{C}}$ acts via any character.
$$\begin{cases} \pi(h)v_m = 2m\, v_m \\ \pi(f)v_m = v_{m-1} \\ \pi(e)v_m = (\ell-m)(\ell+1+m)v_{m+1} \end{cases}$$

② Any f.d. rep of $\mathfrak{g}$ is a sum of irreps

Cor: $\pi(\mathfrak{t}_{\mathbb{C}})$ are jointly diagonalizable in any f.d. rep'n
($\Leftrightarrow$ rep'n is a sum of weight spaces)

Today: Universal enveloping algebra.

(analogue of group ring for Lie algebras)

Def: Fix field $\mathbb{F}$, $\mathbb{F}$-vsp V. The tensor algebra of V is the vsp

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

equipped with the graded algebra structure coming from isomorphisms $V^{\otimes k} \otimes V^{\otimes l} = V^{\otimes (k+l)}$.

Write $\iota : V \to T(V)$ for isom $V \xrightarrow{\cong} V^{\otimes 1}$.

Lemma: Let $B = \{\underline{v}_i\}_{i \in I} \subset V$ be a basis.
For $\sigma : [n] \to I$ write

$$\underline{v}_\sigma = \underline{v}_{\sigma(0)} \otimes \underline{v}_{\sigma(1)} \otimes \cdots \otimes \underline{v}_{\sigma(n-1)}$$

Then $\{\underline{v}_\sigma\}_{\sigma \in I^{[n]}} \subset V^{\otimes n}$ is a basis,

$$\coprod_{n=0}^{\infty} \{\underline{v}_\sigma\}_{\sigma \in I^{[n]}} \subset T(V) \text{ is a linear basis}$$

Prop: The tensor algebra is a unital associative $\mathbb{F}$-algebra, generated by V (in fact by B).
It has the universal property that for any $\mathbb{F}$-alg. A, any $\mathbb{F}$-linear map $f : V \to A$, there is a unique $\mathbb{F}$-alg. hom $\tilde{f} : T(V) \to A$ s.t. $\tilde{f} \circ \iota = f$.

Pf: Given f define $f_n : V^n \to A$ by

$$f_n(v_1, \ldots, v_n) = f(v_1)\cdot f(v_2) \cdot \cdots f(v_n)$$

this is n-linear map $V^n \to A$, so extends uniquely to $\tilde{f}_n : V^{\otimes n} \to A$.

Now by univ. of prop. of $\oplus$ have $\tilde{f} = \bigoplus_n \tilde{f}_n$.

This is an alg. hom by univ. property of $V^{\otimes k} \otimes V^{\otimes l}$
$$\Downarrow$$
$$V^{\otimes(k+l)}.$$

$T(V)$ = ring of non-commutative polynomials in B.

Sf $\mathfrak{g}$ Lie alg., $\pi : \mathfrak{g} \to \mathrm{End}_F(V)$ set extension $\tilde{\pi} : T(\mathfrak{g}) \to \mathrm{End}_F(V)$; if $X, Y \in \mathfrak{g}$

then $\tilde{\pi}(XY - YX) = \tilde{\pi}(X)\tilde{\pi}(Y) - \tilde{\pi}(Y)\tilde{\pi}(X)$ $\tilde{\pi}$ is a alg hom

pdts in $T(\mathfrak{g})$

$$= [\tilde{\pi}(X), \tilde{\pi}(Y)]_{\mathrm{End}_F(V)}$$

$\tilde{\pi}$ extends π $\to$ $= [\pi(X), \pi(Y)]_{\mathrm{End}_F(V)}$

π is a lie alg rep'n $\to$ $= \pi([X, Y])$

$$= \tilde{\pi}([X, Y])$$

Def: $J \subset T(\mathfrak{g})$ be the two-sided ideal generated by $\{XY - YX - [X,Y]\}_{X,Y\in\mathfrak{g}}$

$\cap\ \mathfrak{g}\oplus\mathfrak{g}^{\otimes 2}$

The universal enveloping algebra of $\mathfrak{g}$ is the algebra $\mathcal{U}(\mathfrak{g}) = T(V)/J$.

Write $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ for the composition of inclusion $\mathfrak{g} \to T(\mathfrak{g})$, quotient map

Lemma: $\mathcal{U}(\mathfrak{g})$ is unital (hence $\neq 0$),

$\iota([X,Y]) = \iota(X)\iota(Y) - \iota(Y)\iota(X)$

Pf: J is contained in the maximal ideal $\bigoplus_{n\geq 1} \mathfrak{g}^{\otimes n}$.
second claim true by construction.

Prop: For every assoc. F-alg. A, any Lie alg. hom $f : \mathfrak{g} \to A$ extends uniquely to an F-alg. hom $\tilde{f} : \mathcal{U}(\mathfrak{g}) \to A$

Pf: f extends uniquely to $T(\mathfrak{g})$, extension must factor through J.

Lemma: Let $\pi : \mathfrak{g} \to \mathrm{End}_F(V)$ be a Lie alg. hom, $\underline{v} \in V$. Then $\mathcal{U}(\mathfrak{g})\underline{v}$ is the subrep'n gen. by $\underline{v}$

Pf: If $\mathcal{U} \subset V$ is that subrep'n then $\pi|_{\mathcal{U}}$ is a Lie alg. hom $\mathfrak{g} \to \mathrm{End}_F(\mathcal{U})$ so π acts on $\mathcal{U}$,

$\mathcal{U}(\mathfrak{g}) \cdot \underline{v} \subset \mathcal{U}$.

Also, $\mathcal{U}(\mathfrak{g})$ is left-$\mathfrak{g}$-inv't

so $\mathcal{U}(\mathfrak{g})\underline{v}$ is a subrep'n

Thm: (Poincaré-Birkhoff-Witt) Suppose $\mathrm{char}(F) = 0$. Let $\{X_i\}_{i \in B} \subset \mathfrak{g}$ be an ordered basis

Then $\left\{ X_\sigma \,\middle|\, \begin{matrix} \sigma : [n] \to I \\ \text{nondecreasing} \end{matrix} \right\} \subset \mathcal{U}(\mathfrak{g})$ is a basis

Example: $\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{span}\{f, h, e\}$

then $\{f^i h^j e^k\}_{i,j,k \geq 0}$ is a basis of $\mathcal{U}(\mathfrak{g})$

Pf: (spanning) $T(V)$ has spanning set consisting of all words in f, h, e.

NTS: each such word in $\mathcal{U}(\mathfrak{g})$ is in span of ordered words

Say word is $t \cdot Y \cdot X \cdot s$, t, s words, $Y, X \in B$, $Y > X$

in order. Then in $\mathcal{U}(\mathfrak{g})$ $tYXs = t(XY)s + t[Y,X]s$
$t[Y,X]s$ is a combination of shorter words
by induction is spanned by claimed basis

After finitely many swaps can sort any words.

Pf: (indep) to be added to the notes

Cor: Say $\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i$ where $\mathfrak{g}_i$ are subalgebras
Then
$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_1) \cdot \mathcal{U}(\mathfrak{g}_2) \cdots \mathcal{U}(\mathfrak{g}_r)$$

Back to $\mathfrak{g} = \text{Lie } G$, G cpt ctd.

Defi: Let $\mathfrak{n} \subset \mathfrak{g}_{\mathbb{C}}$ be the subalg. generated by $\{\mathfrak{g}_\alpha\}_{\alpha\in\Delta}$.

Lemma: $\mathfrak{n} = \bigoplus_{\beta\in P} \mathfrak{g}_\beta$. for some subset $\Delta \subset P \subset \Phi^+$;

for each $\alpha\in\Delta$: $\text{ad}_{\mathfrak{g}_{-\alpha}}(\mathfrak{n}) \subset \mathbb{C}\cdot\check{\alpha} \oplus \mathfrak{n}$.

Pf: First claim holds since $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma]$ if nonzero is all of $\mathfrak{g}_{\beta+\gamma}$.

For second claim if $\beta \in P$, $\beta = \sum_\alpha n_\alpha \cdot \alpha$ write $|\beta|_1 = \sum_\alpha n_\alpha$. Consider

$$\mathrm{ad}_{X_{-\alpha}} \cdot X_\beta = [X_{-\alpha}, X_\beta]$$

for $\beta \in P$,

(1) if $\beta = \alpha$, $[X_{-\alpha}, X_\alpha] \propto \alpha^\vee$.

(2) if $\beta \in \Delta \setminus \{\alpha\}$, $[X_{-\alpha}, X_\beta] = 0$ since $\beta - \alpha$ is not a root.

If $\beta \in P$, $|\beta|_1 \geq 2$ then $X_\beta = [X_\delta, X_\gamma]$ where $\delta \in \Delta$, $\gamma \in P$, $|\gamma|_1 = |\beta|_1 - 1 < |\beta|_1$.

So $\mathrm{ad}\, X_{-\alpha} [X_\delta, X_\gamma] = [\underbrace{\mathrm{ad}_{X_{-\alpha}} X_\delta}_{\in \mathbb{C}\alpha^\vee}, X_\gamma] + [X_\delta, \mathrm{ad}_{X_{-\alpha}} X_\gamma]$

$\in \mathfrak{g}_\gamma + [X_\delta, \mathbb{C}\alpha^\vee \oplus \mathfrak{n}] \in \mathfrak{g}_\gamma + \mathfrak{g}_\delta \oplus \mathfrak{n} \subseteq \mathfrak{n}$.

<u>Prop</u>: $\mathfrak{n} = \bigoplus_{\beta > 0} \mathfrak{g}_\beta \quad (P = \Phi^+)$

<u>Pf</u>: Let $\bar{\mathfrak{n}}$ be the complex conjugate,

$$\bar{\mathfrak{n}} = \bigoplus_{\beta \in P} \mathfrak{g}_{-\beta}.$$

<u>Claim</u>: $\bar{\mathfrak{n}} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}$ is a subalgebra

<u>Pf</u>: It is $\mathrm{ad}_{X_{-\alpha}}$-inv't. for each simple root α

By symmetry also $\mathrm{ad}_{X_{-\alpha}}$ inv't, also ad H-inv't, $H \in \mathfrak{t}_{\mathbb{C}}$

But $\{\mathfrak{t}_{\mathbb{C}}\} \cup \{X_{-\alpha}\}_{\alpha\in\Delta} \cup \{X_\alpha\}_{\alpha\in\Delta}$ generate this set.

This Lie algebra is defined over $\mathbb{R}$, say it's $\mathfrak{h}_{\mathbb{C}}$ for $\mathfrak{h} \subset \mathfrak{g}$. $\mathfrak{h}$ has inv't inner prod, so adjoint gp is cpt, inverse image $H < G$ is closed

Δ still system of simple roots of $H \Rightarrow$ same Weyl gp (W generated by $\{s_\alpha\}_{\alpha\in\Delta}$).

$\Rightarrow$ roots of H are W-inv't. But $\Phi = W\cdot\Delta$ so Φ = roots of H, $H = G$.

$\#$

<u>Conclusion:</u> $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) = \mathcal{U}(\bar{\mathfrak{n}})\cdot\mathcal{U}(\mathfrak{t}_{\mathbb{C}})\cdot\mathcal{U}(\mathfrak{n})$

(e.g. $\mathcal{U}(\mathfrak{sl}_2\mathbb{C}) = \mathcal{U}(\mathbb{C}f)\cdot\mathcal{U}(\mathbb{C}h)\cdot\mathcal{U}(\mathbb{C}e)$]

That $\mathfrak{n} = \bigoplus_{\beta>0} \mathfrak{g}_\beta$ means: start with $\{X_\alpha\}_{\alpha\in\Delta}$ repeatedly take commutators get all $\{X_\beta\}_{\beta>0}$

so $\{X_\alpha, X_{-\alpha}\}_{\alpha\in\Delta} \cup \mathfrak{t}_{\mathbb{C}}$ generate $\mathfrak{g}_{\mathbb{C}}$