Math 535, Lecture 8, 25/1/2023

Last time: Manifolds

Chart in M: (U, f) : $U \subset M$ open
$f: U \to \mathbb{R}^n$ homeo onto open set

charts compatible if transition maps smooth

Atlas: cover of M with compatible charts

$f: M \to N$ smooth if smooth in every patch

Example: Let A assoc algebra, $[a,b] = ab - ba$ gives A a Lie algebra structure.

derivation on A: $d \in \mathrm{End}_{k\text{-vsp}}(A)$ s.t.

Leibnitz rule $\to$ $d(ab) = da \cdot b + a \cdot db$

Ex: (1) $D_A = \left\{ \begin{matrix} k\text{-derivations} \\ \text{on } A \end{matrix} \right\} \subset \mathrm{End}_{k\text{-vsp}}(A)$
is a Lie subalgebra

(2) $a \mapsto [a, \cdot]$ is a Lie algebra hom $A \to D_A$.

(1): If $d, d' \in \mathcal{D}_A$ so is $[d, d']$.

$$[d, d'] \cdot a = d(d'a) - d'(da)$$

(2) $[a, [b, c]] = [[a, b], x] = [a, [b, x]] - [b, [a, x]]$.

Today: Vector fields, differentiation, tangent spaces.

Key example: $A = C^\infty(M)$.

Def: A (smooth) vector field on M is a derivation of $C^\infty(M)$. Write D_M for $D_{C^\infty(M)}$

Lemma: Let $X \in D_M$, $f, g \in C^\infty(M)$

(1) If f is constant, $Xf = 0$

(2) If $f(p) = g(p) = 0$ then $X(fg)(p) = 0$

(3) If f is constant in a neighbourhood of p then $(Xf)(p) = 0$

Cor: If $f=g$ in a nbd of p, $(Xf)(p)=(Xg)(p)$

Pf: If $f=1$ then $f^2=f$ then

$$Xf = X(f^2) = 2\,Xf\cdot f = 2\,Xf.$$

If $f(p)=g(p)=0$ then

$$X(fg)(p) = (Xf)(p)\,g(p) + f(p)\,(Xg)(p) = 0$$

Let $p\in U$ with U open, $f|_U = 0$. Choose $g\in C_c^\infty(U)$ s.t. $g(p)\neq 0$ then $fg=0$ and

$$0 = X(fg)(p) = Xf(p)\cdot \underset{\neq 0}{g(p)} + \cancel{f(p)\,Xg(p)}$$

$\Rightarrow Xf(p)=0.$ ∎

Def: $I_p = \{ f\in C^\infty(M) \mid f(p)=0 \}$ $\quad p\in M.$

This is a maximal ideal of $C^\infty(M)$ (Kernel of evaluation at p).

Def: The cotangent space at p is $T_p^*M = I_p/I_p^2$ (∞ $\mathbb{R}$-vsp)

Lemma: (Hadamard) $T_p^* M$ is n-dim ($n = \dim M$) and

$$\bigcup_p T_p^* M \text{ is a smooth vector bundle.}$$

Pf: Let f vanish near p, say in nbd $\mathcal{U}$. Choose $g \in C_c^\infty(\mathcal{U})$ s.t. $g(p) = 1$.

Then $fg = 0$ so $f(1-g) = f$ but $1-g \in I_p$ so $f \in I_p^2$. $\Rightarrow$ if f, g agree on nbd of p then $f \equiv g$ (I_p^2), enough to work locally.

For a co-ordinate patch may assume working in $\underline{0} \in \mathcal{U} \subset \mathbb{R}^n$.

Now given $f \in C^\infty(\mathcal{U})$ set $g(t) = f(t\underline{x})$

Then $f(\underline{x}) - f(0) = g(1) - g(0) = \int_0^1 g'(t)\,dt.$

$$= \int_0^1 \underline{x} \cdot \nabla f(t\underline{x})\,dt = \underline{x} \cdot \int_0^1 \nabla f(t\underline{x})\,dt =$$

$$= \sum_{i=1}^n x_i \cdot \int_0^1 \frac{\partial f}{\partial x^i}(t\underline{x})\,dt = \sum_{i=1}^n \left[\frac{\partial f}{\partial x^i}(0) \cdot x_i + x_i h_i \right]$$

where $h_i = \int_0^1 \left(\frac{\partial f}{\partial x^i}(t\underline{x}) - \frac{\partial f}{\partial x^i}(0) \right) dt$

But $h_i(\underline{0}) = 0$, set $f - \nabla f(0) \cdot \underline{x} \in I_0^2$.

$\Rightarrow$ Every class in I_0 / I_0^2 has a linear rep.

To see they are inequivalent: if f is linear near 0, some directional derivative of f is nonzero at 0, but directional derivatives are derivations on $C^\infty(U)$, so vanish on I_0^2.

$\blacksquare$

(for cotangent bundle see diff. geom. textbook)

<u>Def:</u> The linear dual $T_pM \overset{\text{def}}{=} (T_p^*M)^*$ is called the space tangent to M at p

<u>Lemma:</u> For $X \in D_M$, $f \mapsto (Xf)(p)$ decends to an element of T_pM. The linear map $D_M \to T_pM$ is surjective.

<u>Pf:</u> Need to show $(Xf)(p) = 0$ $\forall f \in I_p^2$
- that was a lemma

for surjectivity work locally, use $a_i(x)\frac{\partial}{\partial x^i}$ where a_i smooth cpt support, $\equiv 1$ near 0. $\blacksquare$

Ex: $T_pM = \{\mathbb{R}$-valued derivations on algebra of germs of smooth fcns at $p\}$

<u>Prop:</u> (1) let $X \in \mathcal{D}_M$, $U \subset M$ open, let $f \in C^\infty(U)$ for $p \in U$ let $h \in C_c^\infty(U)$ st $h \equiv 1$ near p
Set
$$(X\restriction_U f)(p) = X(fh)(p).$$

Then $X\restriction_U$ is a derivation on $C^\infty(U)$, agrees with X when it should (if $f \in C^\infty(M)$ then $Xf\restriction_U = X\restriction_U \cdot f\restriction_U$) and map $\mathcal{D}_M \to \mathcal{D}_U$ is a map of Lie algebras

(2) Let $\{U_i\}_{i\in B}$ be an open cover of M, $X, Y \in \mathcal{D}_M$. Suppose $X\restriction_{U_i} = Y\restriction_{U_i}$. Then $X = Y$

(3) Let $\{U_i\}_{i\in B}$ be an open cover, $X_i \in \mathcal{D}_{U_i}$. Suppose for all i,j $X_i\restriction_{U_i\cap U_j} = X_j\restriction_{X_i \cap X_j}$.

Then $\exists X \in \mathcal{D}_M$ st. $X\restriction_{U_i} = X_i$.