Math 535, Lecture 7, 23/1/2023

<u>Last time</u>: Peter-Weyl thm: G cpt sp

① $L^2(G) = \widehat{\bigoplus_{\sigma \in \hat{G}}} \mathcal{C}(\sigma)$

(initially sum was over f.d. irreps)

<u>Cor</u>: $\bigoplus_{\sigma \in \hat{G}} \mathcal{C}(\sigma) = C(G)_G = L^2(G)_G$ is dense in $C(G)$

② For any cts rep'n on reasonable TVS V, V_G is dense in V.

<u>Cor</u>: Every irrep of G is f.d.

$\hat{G}$ = unitary dual = $\left\{\begin{array}{l}\text{isom classes of irred} \\ \text{unitary rep'ns}\end{array}\right\}$

unitary rep'n = vsp is a Hilbert space, group action is unitary.

<u>Today</u>: Manifolds & Lie Groups

(II) Lie groups & Lie algebras

① Manifolds

Notation: $U \subset \mathbb{R}^n$ open, write $C^\infty(U;\mathbb{R}^m)$ for the smooth (= ∞'ly diff) $\mathbb{R}^m$-valued fcns on U.

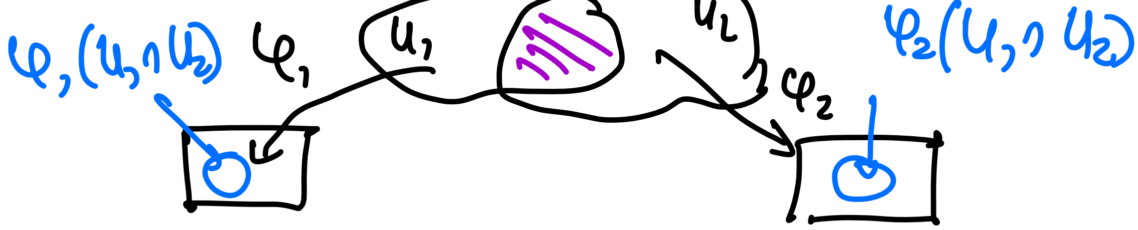
Def: A coordinate chart (or patch) in a top space M is a pair (U,φ) where $U \subset M$ is open, $\varphi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$.

[co-ordinates of $\mathbb{R}^n$ become co-ordinate functions on U by composition with φ]

Two charts (U_1,φ_1), (U_2,φ_2) are compatible if

$$\varphi_1\restriction_{U_1\cap U_2} \circ \varphi_2\restriction_{U_1\cap U_2}^{-1}, \quad \varphi_2\restriction_{U_1\cap U_2} \circ \varphi_1\restriction_{U_1\cap U_2}^{-1}$$

are smooth



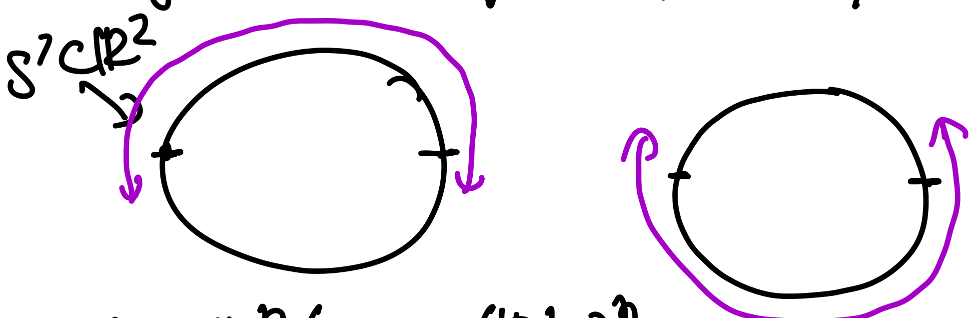
Def: An atlas on M is a covering of M by compatible charts. A smooth manifold is a pair (M, A) where M is a σ-cpt top space and A is an atlas on M.

for id_M

Ex ① Every atlas is contained in a maximal atlas
② Two atlases define diffeomorphic manifolds iff they are contained in the same maximal atlas
(maximal atlas also called "smooth structure" on M)

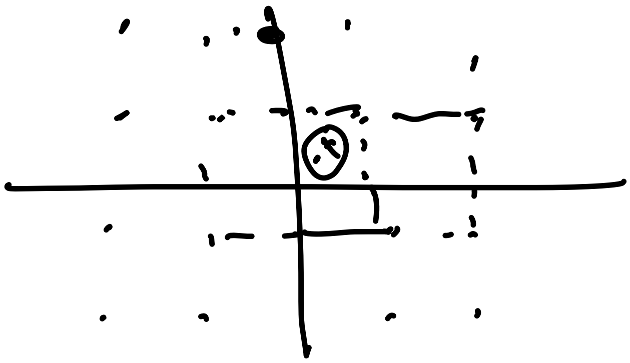
Example: (0) $\mathbb{R}^n$ (only 1 chart needed)
(1) Any open $U \subset \mathbb{R}^n$.
(includes $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$)

(2) $S^n = \{ \underline{x} \in \mathbb{R}^{n+1} \mid |\underline{x}| = 1 \}$
① can take open hemispheres
② can spherical caps beyond equator:



③ $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$

point $\underline{x} \in \mathbb{R}^n$ take U nbd of $\underline{x}$ small enough s.t. translates $\{U + \gamma\}_{\gamma \in \mathbb{Z}^n}$ disjoint



then quotient map $U \to \mathbb{R}^n/\mathbb{Z}^n$ has an inverse, that's a patch

Fact: (Invariance of domain) If $n \neq m$ then $\mathbb{R}^n, \mathbb{R}^m$ not locally homeomorphic.

$\Rightarrow$ If M ctd manifold all charts have same # of co-ordinates

Def: Let M^m, N^n be smooth manifolds. A cts map $f: M \to N$ is smooth if for any charts $(U, \varphi) \subset M$, $(V, \psi) \subset N$, the map

$$\psi \circ f \circ \varphi^{-1}$$

is smooth $\varphi(U \cap f^{-1}(V)) \longrightarrow \psi(V)$.

Ex: (1) id_M is smooth; (2) composition of smooth maps is smooth.

<u>Remarks</u> Implicit fcn thm shows that level sets of smooth fcns are often manifolds (eg. $x^2+y^2+z^2=1$).

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② Tangent & cotangent spaces.

<u>Defs</u> A Lie algebra over a field k is a pair $(\mathfrak{g}, [\cdot,\cdot])$ where $\mathfrak{g}$ is a k-vsp and

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is a bilinear form s.t.

(1) (alternativity) $[X,X]=0$

(2) Jacobi identity $[[X,Y],Z] + \text{cyc} = 0$

(<u>Ex</u>. alternating $\Rightarrow$ antisymmetric $[Y,X] = -[X,Y]$ reverse true if char $k \neq 2$)

<u>Examples</u>: A associative k-algebra

① A itself, equipped with $[a,b] = ab - ba$.

② Call $d \in \mathrm{End}_{k\text{-vsp}}(A)$ a derivation if $d(ab) = da \cdot b + a \cdot db$

then $\mathcal{D}_A = \left\{ \begin{matrix} \text{derivations} \\ \text{on } A \end{matrix} \right\}$ is a Lie algebra,

a subalgebra of $\mathrm{End}_{k\text{-vsp}}(A)$

(map $d_a(b) = [a, b]$ is a Lie alg hom

$$A \to \mathcal{D}_A$$

image = "inner derivations"

kernel = $Z(A)$.)