Math 535, lecture 6 20/1/2023

Last time: Defined G-finite vectors in a rep'n: in (σ, V)

$$V_G = \{ v \in V \mid \mathrm{Span}_{\mathbb{C}}\{\pi(g)v \mid g \in G\} \text{ f.d.} \}$$

Saw $V_G \subset V$ is a G-inv't alg. subspace.

G cpt: $\bigoplus_{\substack{\sigma \text{ f.d.} \\ \text{irrep}}}^{\text{in}} \mathcal{E}(\sigma) \subseteq C(G)_G \subsetneq L^2(G)_G$ have equality (if $W \subset L^2(G)_G$ irrep then $W \subset \mathcal{E}(W)$).

Today: ① $C(G)_G = L^2(G)_G$ is dense in both spaces.

② V_G is dense in V for all rep'ns

Tool: continuous group algebra $C(G)$ (G cpt)

If $f \in C(G)$, π rep'n define $\pi(f) = \int_G f(g)\,\pi(g)\,dg$

Ex: Define $(\psi * \varphi)(x) = \int_{yz=x} \psi(y)\varphi(z)\,dy =$

$$= \int \psi(y)\,\varphi(y^{-1}x)\,dy = \int \psi(xz^{-1})\varphi(z)\,dz$$

(convolution of f, g)

Then $\pi(\psi)\,\pi(\varphi) = \pi(\psi * \varphi)$.

$\pi(a\psi + \varphi) = a\,\pi(\psi) + \pi(\varphi)$

Observe: $\pi(\psi)\,\pi(h) = \int_G \psi(g)\,\pi(gh)\,dg$ ($g \in G$)

$$= \int_G \psi(gh^{-1})\,dg = \pi(R_{h^{-1}}\psi)$$

$$\pi(h)\pi(\psi) = \pi(L_h \psi)$$

Cor: $\{\pi(h)\,\pi(\psi)\,\underline{v}\}_h = \{\pi(L_h\psi)\underline{v}\}$

So if $\underline{\psi} \in C(G)_G$ then $\pi(\psi)\underline{v}$ is G-finite

Goal: $C(G)_G$ dense in $C(G)$ so V_G is dense in V

Theorem: (Peter-Weyl I) $L^2(G) = \hat{\bigoplus}_{\pi} \mathcal{C}(\pi)$

f.d. irreps

Pf: We know (Schur orthogonality) that the sum $\sum \mathcal{C}(\sigma)$ is orthogonal, need to show it's dense

Let $V = \left(\bigoplus_{\pi} \mathcal{C}(\pi)\right)^{\perp}$.

Which is a sub rep'n of $(L^2(G), R)$.

Suppose have $f \in V$ non-zero.

Build $\psi \in C(G)$ s.t. $L(\psi) f \neq 0$, s.t. $L(\psi)$ is s.a. & cpt acting on $L^2(G)$ hence on V. Then $L(\psi)|_V$ is s.a., cpt, has f.d. eigenspaces. Since left, right actions commute each eigenspace is R-inv't, so V will contain G-finite vectors $\Rightarrow \Leftarrow$

Action of G on $L^2(G)$ cts $\Rightarrow$ have nbd $\mathcal{U}$ of e s.t. if $u \in \mathcal{U}$ then $\|L(u) f - f\| \leq \frac{1}{2}$, replace $\mathcal{U}$ with $U \cap U^{-1}$

Let $\chi \in C_c(\mathcal{U})$ s.t. $\chi(u) \geq 0$, $\chi(x) = \chi(u^{-1})$

normalize st $\int_G \chi \, d\mu = 1$

Then since ball $B_{L^2(G)}(f, \frac{1}{2})$ is convex,

$\| L(\chi) f - f \| \le \frac{1}{2}$ so $L(\chi) f \neq 0$

(so $L(\chi)|_V \neq 0$)

<u>Ex</u>: If π unitary, $\pi(\psi)^+ = \pi(\check{\psi})$
where $\check{\psi}(g) = \overline{\psi(g^{-1})}$ (use $\pi(g)^+ = \pi(g)^{-1} = \pi(g^{-1})$)

so by choice of χ, $L(\chi)^+ = L(\chi)$.

Also on cpt gp, $L(\psi)$ is Hilbert-Schmidt
since its kernel is $\psi(xy^{-1})$

$$(L(\psi) f)(x) = \int \psi(y) f(y^{-1}x) \, dy =$$

$$= \int_G \psi(xy^{-1}) f(y) \, dy$$

since $G \times G$ is cpt, kernel is in $L^2(G \times G)$
so $L(\psi)$ is Hilbert-Schmidt, hence compact. ∎

Cor: (Peter-Weyl II) $C(G)_G = \bigoplus_{\pi} \mathcal{C}(\pi)$ is dense in $C(G)$

Pf: HW: if Φ_1, Φ_2 matrix coeff of π, σ then $a\Phi_1 + \Phi_2$ matrix coeff of $\pi \oplus \sigma$

$\Phi_1 \cdot \Phi_2$ " " " $\pi \otimes \sigma$

$\overline{\Phi_1}$ is a matrix coeff of $\check{\pi}$.

So $\bigoplus_{\pi} \mathcal{C}(\pi)$ is a subalgebra of $C(G)$

closed under $f \mapsto \bar{f}$. Also $\mathbb{1}$ = matrix coeff of triv rep'n. Wts: algebra separates pts

Since algebra is G-inv't enough to separate $g \in G$ from id.

Observe: $\bigcap_{\pi} \mathrm{Ker}(\pi)$ fixes all π so all $\mathcal{C}(\pi)$ so all of $L^2(G)$, so is trivial

So have π st. $\pi(g) \neq id$ (so $\pi \not\equiv id$)

$\Rightarrow \exists\, \underline{v} \in V_\pi$ st. $\pi(g)\underline{v} \neq \underline{v}$ wlog $\|\underline{v}\| = 1$
then $\|\pi(g)\underline{v}\| = 1$

then $\langle \underline{v}, \pi(g)\underline{v} \rangle \neq 1$

i.e. $\Phi^{\pi}_{v,v}(g) \neq \Phi^{\pi}_{v,v}(e)$.

By Stone-Weierstraß $\bigoplus_{\pi} \mathcal{C}(\pi)$ is dense.

$\blacksquare$

Theorem: (Peter-Weyl II) Every irrep of G is f.d., for any rep'n V_G is dense in V.

Pf: Second claim follows from density of $C(G)_G$ in $C(G)$:

for any $\underline{v} \in V$ have $\psi \in C(G)$ st: $\pi(\psi)\underline{v}$ is arbitrarily close to $\underline{v}$ (ψ supported near e)
Now approximate ψ by G-finite fcn

First claim: If (σ, V) irrep, the G-finite fcns are dense so V contains $\neq 0$ G-inv't f.d. subspace.