Math 535, lecture 4, 16/1/2023

Last time: rep'ns of locally cpt groups
matrix coefficients

Today: Compact groups ① f.d. rep'ns
② ∞ dim : Peter-Weyl

Fix a cpt gp G, equipped with a probability Haar measure dg.

(E.g. G finite then $\int_G f(g)\,dg = \frac{1}{\# G} \sum_{g \in G} f(g)$)

① Finite dim rep'ns

Lemma: let (π, V) be a f.d. rep'n of G.
Then $\exists$ a G-inv't Hermitian pdt on V.
("every f.d. rep'n is unitary")

Pf: let $(\cdot,\cdot)$ be any Hermitian pdt,

Set $\langle u, v \rangle = \int_G (\pi(g)u, \pi(g)v)\,dg$

cts fcn on G.

Clearly additive, linear in 2^{nd} co-ord, Hermitian-symmetric. Positive: if $\underline{v} = \underline{v} \neq \underline{0}$ then $g \mapsto (\pi(g)\underline{v}, \pi(g)\underline{v})$ is a positive cts fcn on G so has positive Integral

Finally $\langle \pi(g_0)\underline{v}, \pi(g_0)\underline{v} \rangle = \int_G \left(\pi(g)\pi(g_0)\underline{v}, \pi(g)\pi(g_0)\underline{v}\right) dg$

$= \int_G \langle \pi(gg_0)\underline{v}, \pi(gg_0)\underline{v}) \, dg = \int_G \langle \pi(g)\underline{v}, \pi(g)\underline{v}) dg$

$= \langle \underline{v}, \underline{v} \rangle$ for all $g_0 \in G$.

$\blacksquare$

<u>Cor</u>: Let $W \subset V$ be a G-inv't subspace Then $W^\perp$ [wrt $\langle \cdot, \cdot \rangle$] is G-inv't.

$\Rightarrow$ Every subrep'n has a complement.

<u>Thm</u>; (Maschke) Every f.d. rep'n of G is a direct sum of irreducibles

<u>Pf</u>: Let (π, V) be a f.d. rep'n, $W \subset V$ be max'l wrt inclusion among all direct sums of irreps. If $W \neq V$ then $W^\perp \neq \{0\}$ and then a minimal inv't subspace in $W^\perp$ is a

irrep, can be added to W. ⇒⇐ ∎

what about uniqueness?

<u>Goal:</u> If we write $V = \bigoplus_{\sigma \text{ irred}} m(\sigma)\cdot\sigma$

then $m(\sigma)$ uniquely defined, on each σ unique inner prod up to scaling.

<u>Prop:</u> (Schur's lemma) let (π, V), $(\sigma, W) \in \mathrm{Rep}(G)$ be f.d. irreps. Then

$$\mathrm{Hom}_G(V,W) \cong \begin{cases} \mathbb{C} & \pi \cong \sigma \\ 0 & \pi \not\cong \sigma \end{cases}$$

<u>Proof:</u> If $T \in \mathrm{Hom}_G(V,W)$

the $\mathrm{Ker}\, T \subset V$, $\mathrm{Im}(T) \subset W$ are G-subspaces

If $T \neq 0$, V irred $\Rightarrow \mathrm{Ker}\, T = \{0\}$

W irred $\Rightarrow \mathrm{Im}\, T = W$

so if V, W irred $T \neq 0$ then T is an isom

If $(\pi, V) = (\sigma, W)$ get that $\mathrm{End}_G(V,V)$ is a division algebra. Only f.d. division algebra/$\mathbb{C}$ is $\mathbb{C}$. ∎

Bring in matrix coefficients. Observe: G cpt so matrix coeffs are L^2.

<u>Prop:</u> Let $\pi, \sigma \in \mathrm{Rep}(G)$ be f.d. irreps

① If $\pi \not\cong \sigma$ any two matrix coeffs on π, σ are $\perp$ in $L^2(G)$

② Let $d_\pi = \dim V_\pi$. Then for $u, v \in V_\pi$, $u', v' \in V_\pi'$

we have

$$\langle \phi^\pi_{u,u'}, \phi^\pi_{v,v'} \rangle = \frac{1}{d_\pi} \langle u', v \rangle \langle v', u \rangle$$

<u>Pf:</u> For any $T: V \to W = W_\sigma$ a linear map set

$$\bar{T} = \int_G \sigma(g)^{-1} T \cdot \pi(g)\, dg$$

Then
$$\bar{T} \cdot \pi(g_0) = \int_G \sigma(g)^{-1} T\, \pi(g g_0)\, dg$$
$$\underset{g g_0 \mapsto g}{=} \int_G \sigma(g g_0^{-1})^{-1} T\, \pi(g)\, dg$$
$$= \sigma(g_0) \bar{T}.$$
so $\bar{T} \in \mathrm{Hom}_G(V, W)$

Now given $\underline{v} \in V$, $\underline{v}' \in V'$, $\underline{w} \in W$, $\underline{w}' \in W'$, set

$$T = |w\rangle\langle v'|.$$

Then

$$\langle \underline{w}' | \bar{T} | \underline{v} \rangle = \int_G \langle \underline{w}' | \sigma(g^{-1}) | \underline{w} \rangle \langle \underline{v}' | \pi(g) | \underline{v} \rangle$$

$$= \int_G \overline{\langle \underline{w} | \sigma(g) | \underline{w}' \rangle} \langle \underline{v}' | \pi(g) | \underline{v} \rangle \, dg$$

$$= \langle \Phi^{\sigma}_{\underline{w}', \underline{w}}, \Phi^{\pi}_{\underline{v}, \underline{v}'} \rangle_{L^2(G)}$$

If $V \neq W$ then $\bar{T} = 0$ so inner pdt $= 0$

If $V = W$ then $\bar{T} = \lambda \cdot id$.

$$\bar{T} = \int_G \pi(g)^{-1} T \pi(g) \, dg \quad \text{so} \quad \operatorname{Tr} \bar{T} = \operatorname{Tr} T$$

$$\Rightarrow \lambda = \frac{1}{d_\pi} \cdot \operatorname{Tr} T$$

If $T = |w\rangle\langle v'|$ then $\operatorname{Tr} T = \operatorname{Tr} \langle \underline{v}' | \underline{w} \rangle$.

so $\bar{T} = \frac{1}{d_\pi} \langle \underline{v}', \underline{w} \rangle$ Then

$$\langle \Phi^\pi_{\underline{w}',\underline{w}}, \Phi^\pi_{\underline{v},\underline{v}'} \rangle = \frac{1}{d_\pi} \langle \underline{w}' | \underline{v} \rangle \langle \underline{v}' | \underline{w} \rangle.$$

Def: The character χ_π is the fcn

$$\chi_\pi(g) = \mathrm{Tr}\, \pi(g) = \sum_{\underline{v} \in \text{basis}} \langle \underline{v} | \pi(g) | \underline{v} \rangle$$

which is a sum of matrix coeff.

Cor If $\pi \neq \sigma$ are irred, $\langle \chi_\pi, \chi_\sigma \rangle_{C(G)} = 0$

If $\pi = \sigma$ $\langle \chi_\pi, \chi_\sigma \rangle = 1$ (sum over ons

$\Rightarrow$ If $\pi \cong \bigoplus_\sigma m(\sigma) \cdot \sigma$ σ distinct irreps

Then $\chi_\pi = \sum_\sigma m(\sigma) \chi_\sigma$

so $\langle \chi_\pi, \chi_\sigma \rangle = m(\sigma) \Rightarrow m(\sigma)$ unique

Cor: For irrep σ write $C(\sigma)$ = space of matrix coeff of σ.

Then $\bigoplus_{\text{f.d. irreps } \sigma} \mathcal{E}(\sigma) \subseteq L^2(G)$ is an orthogonal sum $\blacksquare$

Let $SL_2(\mathbb{R})$ act on $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}$
s.t. $f(r\underline{x}) = r^s f(\underline{x})$
$s \in \mathbb{C}$.

Write $|\underline{v}\rangle$ for $\underline{v} \in V$

$\langle \underline{v}'|$ for $\underline{v}' \in V'$

write $\langle \underline{v}' | \underline{v} \rangle$ for $\underline{v}'(\underline{v})$

(if V Hilbert space for each $\underline{v} \in V$
have functional
$\langle \underline{v} | \underline{w} \rangle = \langle \underline{v}, \underline{w} \rangle_V$)

$T = |\underline{v}\rangle\langle \underline{w}'|$ is the linear map

$T(\underline{x}) = |\underline{v}\rangle\langle \underline{w}' | \underline{x} \rangle = \underline{w}'(\underline{x}) \cdot \underline{v}$

$|\underline{v}\rangle, |\underline{w}\rangle \in \mathbb{R}^n$ wrt std inner prod

$$\langle \underline{v}| = \underline{v}^T = (v_1, v_2, \ldots, v_n)$$

$$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\langle \alpha \underline{v} + \underline{w}| = \alpha^* \langle \underline{v}| + \langle \underline{w}|$$

in Hilbert space, where $\langle \underline{v}|$ means "inner prod with $\underline{v}$".
