Lior Silberman's Math 412: Problem Set 3 (due 3/10/2023)

Practice

M1 Let $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as vectors in $\mathbb{R}^3$.

(a) Construct an explicit linear functional $\varphi \in \left(\mathbb{R}^3\right)'$ vanishing on $\underline{u}_1, \underline{u}_2$.

(b) Show that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is a basis on $\mathbb{R}^3$ and find its dual basis.

(c) Evaluate the dual basis at $\underline{u}$.

M2 Let V be n-dimensional and let $\{\varphi_i\}_{i=1}^m \in V'$.

(a) Show that if $m < n$ there is a non-zero $\underline{v} \in V$ such that $\varphi_i(\underline{v}) = 0$ for all i. Interpret this as a statement about linear equations.

(b) When is it true that for each $\underline{x} \in F^m$ there is $\underline{v} \in V$ such that for all i, $\varphi_i(\underline{v}) = x_i$?

M3 Let U, V be finite-dimensional vector spaces and let $L \in \mathrm{Hom}_F(U,V)$. Consider the pairing $V' \times U \to F$ given by $\langle \varphi, \underline{u} \rangle_L = \varphi(L\underline{u})$. Let $\{\underline{u}_j\} \subset U$, $\{\underline{v}_i\} \subset V$ be bases and let $\{\varphi_i\} \subset V'$ be the basis dual to $\{\underline{v}_i\}$. Show that the matrix of L as a linear map $U \to V$ is the same as the Gram matrix of the pairing $\langle \cdot, \cdot \rangle_L$.

Example of linear functionals: Banach limits

Let $\ell^\infty \subset \mathbb{R}^\mathbb{N}$ denote the set of *bounded* sequences (the sequences $\underline{a}$ such that for some M we have $|a_i| \le M$ for all i). Let $S\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ be the *shift* map $(S\underline{a})_n = \underline{a}_{n+1}$. A subspace $U \subset \mathbb{R}^\mathbb{N}$ is *shift-invariant* if $S(U) \subset U$. If U is shift-invariant a function F with domain U is called *shift-invariant* if $F \circ S = F$ (example: the subset $c \subset \mathbb{R}^\mathbb{N}$ of convergent sequences is a shift-invariant subspace, as is the functional $\lim\colon c \to \mathbb{R}$ assigning to every sequence its limit).

Note that M4 is a practice problem!

M4 (Useful facts)

(a) Show that ℓ^∞ is a subspace of $\mathbb{R}^\mathbb{N}$.

(b) Show that $S\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is linear and that $S(\ell^\infty) = \ell^\infty$.

(c) Let $U \subset \mathbb{R}^\mathbb{N}$ be a shift-invariant subspace. Show that the set $U_0 = \{S\underline{a} - \underline{a} \mid \underline{a} \in U\}$ is a subspace of U.

(d) In the case $U = \mathbb{R}^{\oplus\mathbb{N}}$ of sequences of finite support, show that $U_0 = U$.

(e) Let Z be an auxiliary vector space. Show that $F \in \mathrm{Hom}(U,Z)$ is shift-invariant iff F vanishes on U_0.

1. Let $W = \ell_0^\infty \subset \ell^\infty$ in the notation of M4(c). Let $\mathbb{1}$ be the sequences everywhere equal to 1.

(a) Show that the sum $W + \mathbb{R}\mathbb{1} \subset \ell^\infty$ is direct and construct an S-invariant functional $\varphi\colon \ell^\infty \to \mathbb{R}$ such that $\varphi(\mathbb{1}) = 1$ (*Hint*: PS2 problem 5(b)).

(b) (Strengthening) For $\underline{a} \in \ell^\infty$ set $\|\underline{a}\|_\infty = \sup_n |a_n|$. Show that if $\underline{a} \in W$ and $x \in \mathbb{R}$ then $\|\underline{a} + x\mathbb{1}\|_\infty \ge |x|$. (Hint: consider the average of the first N entries of the vector $\underline{a} + x\mathbb{1}$).

SUPP Let $\varphi \in (\ell^\infty)'$ be shift-invariant, positive (if $a_i \ge 0$ for all i then $\varphi(\underline{a}) \ge 0$), and satisfy $\varphi(\mathbb{1}) = 1$. Show that $\liminf_{n\to\infty} a_n \le \varphi(\underline{a}) \le \limsup_{n\to\infty} a_n$ and conclude that the restriction of φ to c is the usual limit.

2. ("choose one") Let $\varphi \in (\ell^\infty)'$ satisfy $\varphi(\mathbb{1}) = 1$. Let $\underline{a}$ be the sequence $a_n = \frac{1+(-1)^n}{2}$.
   (a) Suppose that φ is shift-invariant. Show that $\varphi(\underline{a}) = \frac{1}{2}$.
   (b) Suppose that φ respects pointwise multiplication (if $z_n = x_n y_n$ then $\varphi(\underline{z}) = \varphi(\underline{x})\varphi(\underline{y})$). Show that $\varphi(\underline{a}) \in \{0,1\}$.

SUPP If $U \subset \mathbb{R}^{\mathbb{N}}$ then $\Sigma \in U'$ is called a *summation method* if U contains the convergent series and if Σ extends the usual functional from real analysis. We call Σ *shift-invariant* if its domain is shift-invariant and if $\Sigma(\underline{a}) = a_1 + \Sigma(S\underline{a})$.
   (a) Suppose that the geometric sequence $(x^n)_{n\geq 0}$ ($x \neq 1$) is in the domain of some shift-invariant summation method. Show that $\Sigma\left((x^n)_{n=0}^{\infty}\right) = \frac{1}{1-x}$.
   (b) Consider next the alternating series $a_n = (-1)^{n-1}n$ ($n \geq 1$) . Calculate $\underline{a} + S\underline{a} = (I+S)\underline{a}$ and show that $(I+S)^2\underline{a}$ has finite support. Use that to prove that if the shift-invariant summation method Σ applies to this sequence then

$$\Sigma(1-2+3-4+\dots) = \frac{1}{4}.$$

   (c) Consider next the sum $1+2+3+4+\cdots$. Let $b_n = n$ and let $c_n = \begin{cases} k & n = 2k \\ 0 & n = 2k+1 \end{cases}$. Show that $\underline{b} - 4\underline{c} = \underline{a}$ and conclude that if a shift-invariant summation method can sum $\underline{a}, \underline{b}, \underline{c}$ and has $\Sigma(\underline{b}) = \Sigma(\underline{c})$ (that is $1+2+3+4+\cdots = 0+1+0+2+0+3+\cdots$) then $\Sigma(\underline{b}) = -\frac{1}{12}$.

Duality and bilinear forms

3. Let $F^{\oplus\mathbb{N}}$ denote the space of sequences of finite support. Construct a non-degenerate pairing $F^{\oplus\mathbb{N}} \times F^{\mathbb{N}} \to F$, giving a concrete realization of $\left(F^{\oplus\mathbb{N}}\right)'$.

DEFINITION. Let $T \in \mathrm{Hom}_F(U,V)$ be a linear map. Define a map $T' \colon V' \to U'$ (reverse direction!) by $(T'\varphi)(\underline{u}) = \varphi(T\underline{u})$ for each $\varphi \in V'$.

In class we will check that T' is linear. In the next problem set we will check that the matrix of T' in appropriate bases is the transpose of the matrix fo T, and that $(TS)' = S'T'$ when TS are composable, which will also

4. Let $C_c^\infty(\mathbb{R})$ be the space of compactly supported smooth functions on $\mathbb{R}$ (that is, functions which have derivatives of all orders and which are identically zero outside some interval), and let $D \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ be the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$. For a reasonable function f on $\mathbb{R}$ define a functional φ_f on $C_c^\infty(\mathbb{R})$ by $\varphi_f(g) = \int_{\mathbb{R}} fg\,\mathrm{d}x$ (note that f need only be integrable, not continuous).
   (a) Show that if f is continuously differentiable then $D'\varphi_f = \varphi_{-Df} = -\varphi_{Df}$. (*Hint*: this encodes a basic fact from calculus)

   DEF For this reason one usually *extends* the operator D to the dual space by $D\varphi \stackrel{\text{def}}{=} -D'\varphi$, thus giving a notion of a "derivative" for non-differentiable and even discontinuous functions.
   (b) Let the "Dirac delta" $\delta \in C_c^\infty(\mathbb{R})'$ be the evaluation functional $\delta(f) = f(0)$. Express $(D\delta)(f)$ in terms of f.
   (c) Let φ be a linear functional such that $D'\varphi = 0$. Show that for some constant c, $\varphi = \varphi_{c\mathbb{1}}$.

Supplement

A. Let V be a vector space of positive dimension over an infinite field. Let $\{V_i\}_{i=1}^{r}$ be a finite set of subspaces of V. Show that if $\bigcup_{i=1}^{r} V_i = V$ then $V_i = V$ for some i. In other words, a vector space is not a finite union of proper subspaces.

B. (This is a mostly a problem in analysis) Let $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R})'$.
DEF Let $U \subset \mathbb{R}$ be open. Say that φ is *supported away from* U if for any $f \in C_{\mathrm{c}}^{\infty}(U)$, $\varphi(f) = 0$.
The *support* $\mathrm{supp}(\varphi)$ is the complement the union of all such U.
(a) Show that $\mathrm{supp}(\varphi)$ is closed, and that φ is supported away from $\mathbb{R} \setminus \mathrm{supp}(\varphi)$.
(b) Show that $\mathrm{supp}(\delta) = \{0\}$ (see problem 4(b)).
(c) Show that $\mathrm{supp}(D\varphi) \subset \mathrm{supp}(\varphi)$ (note that this is well-known for functions).
(d) Show that $D\delta$ is not of the form φ_f for any function f.
(e) Find a (discontinuous) function θ such that $D\varphi_\theta = \delta$ as functionals on $C_{\mathrm{c}}^{\infty}(\mathbb{R})$.