Math 100C – SOLUTIONS TO WORKSHEET 9
EULER'S METHOD

1. Compound interest (Bernoulli 1683)

(1) Suppose you have a \$100 bank balance which earns an annual interest rate of 30%.
  (a) Suppose the interest is paid once, at the end of the year. How much would your balance be at that time?
  Solution: $\$100 + 0.3 \times \$100 = \$100 \cdot (1 + 0.3) = \$130$.
  (b) Suppose instead that interest is paid four times a year. What is the quarterly interest *rate*? What would the balance be at the end of the first quarter?
  Solution: The interest reate is $\frac{30\%}{4} = 7.5\%$ so the balance would be $\$100 + 0.075 \cdot \$100 = 107.5$.
  (c) Suppose further that interest is *compounded*: after every quarter the interest is added to the balance. What would be the balance at the end of the year?
  Solution: $\$100 \cdot (1.075)^4 \approx \$133.55$.
  (d) Suppose instead that interest is compounded *daily* and that at a particular day the balance is y dollars. What is the balance the next day?
  Solution: $y(t+1) = y(t) + \frac{30\%}{365} y(t)$

(2) Suppose interest is compounded *continuously* and that at a particular time y the balance is $y(t)$ dollars, where t is measured in years.
  (a) What is the approximate interest rate for the period between times $t, t+h$ if h is very small?
  Solution: $30\% \cdot h$.
  (b) What is the balance at time $t+h$?
  Solution: $y(t+h) \approx y(t) + 30\% \cdot h \cdot y(t)$.
  - Rearranging and taking the limit $t \to 0$ we obtain the ODE $y'(t) = 30\% y(t)$. In general if the interest rate is r we discover that $y(t) = y(0)e^{rt}$.

2. Further examples

From now on let the interest rate by r.

(3) Suppose that in addition to the interest we also have a constant income stream of b dollars per month.
  (a) What differential equation expresses our bank balance now?
  Solution: $y' = ry + b$.
  (b) What is the general solution (hint: use an ansatz of the form $Ce^{rt} + B$). What is the solution that has $y(0) = y_0$?
  Solution: If $y = Ae^{rt} + B$ then $y' = rAe^{rt}$. Thus our ansatz will satisfy the equation iff

$$rAe^{rt} = r(Ae^{rt} + B) + b$$

  or equivalently if

$$rAe^{rt} = rAe^{rt} + (rB + b)$$

  that is if $rB + b = 0$. The general solution is thus $y(t) = Ae^{rt} - \frac{b}{r}$. In particular $y(0) = A - \frac{b}{r}$ so $A = y_0 + \frac{b}{r}$ and the solution is

$$y(t) = \left(y_0 + \frac{b}{r}\right) e^{rt} - \frac{b}{r}.$$

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(4) Suppose instead that our income stream is seasonal, so that the differential equation is $y' = ry + b\sin(2\pi t)$. Find the general solution and the solution satisfying $y(0) = y_0$ using an Ansatz of the form $Ae^{rt} + B\sin(2\pi t) + C\cos(2\pi t)$.

Solution: If $y = Ae^{rt} + B\sin(2\pi t) + C\cos(2\pi t)$ then $y' = rAe^{rt} + 2\pi B\cos(2\pi t) - C\sin(2\pi t)$. Thus our ansatz will satisfy the equation iff

$$rAe^{rt} + 2\pi B\cos(2\pi t) - C\sin(2\pi t) = r\left(Ae^{rt} + B\sin(2\pi t) + C\cos(2\pi t)\right) + b\sin(2\pi t)$$

that is iff

$$rAe^{rt} + 2\pi B\cos(2\pi t) - C\sin(2\pi t) = rAe^{rt} + (rB + b)\sin(2\pi t) + rC\cos(2\pi t)\,,$$

that is if

$$2\pi B\cos(2\pi t) - C\sin(2\pi t) = (rB + b)\sin(2\pi t) + rC\cos(2\pi t)\,.$$

For this to be true we need $2\pi B = rC$ and $rB + b = -C$. Multiplying the second equation by r we get $r^2B + br = -2\pi B$ so $B = -\frac{br}{r^2+2\pi}$ and $C = \frac{2\pi b}{r^2+2\pi}$ – that is

$$y = Ae^{rt} - \frac{br}{r^2 + 2\pi}\sin(2\pi t) - \frac{2\pi b}{r^2 + 2\pi}\cos(2\pi t)\,.$$

Since $y(0) = A - \frac{2\pi b}{r^2+2\pi}$ we see that $A = y_0 + \frac{2\pi b}{r^2+2\pi}$ so the solution is

$$y(t) = \left(y_0 + \frac{2\pi b}{r^2 + 2\pi}\right)e^{rt} - \frac{br}{r^2 + 2\pi}\sin(2\pi t) - \frac{2\pi b}{r^2 + 2\pi}\cos(2\pi t)\,.$$

(5) (For numerical discussion) Suppose instead the *interest rate* is seasonal, so the equation is $y' = (r + a\cos(2\pi t))y$. Can you find a solution? What if $y' = (r + a\sin(2\pi t))y + b$?

Solution: The first equation can be solved by rewriting it as $\frac{y'}{y} = r + a\cos(2\pi t)$ and noting that $\frac{y'}{y} = (\log y)'$. Since $rt + \frac{a}{2\pi}\sin(2\pi t) + C$ has the required derivative we see that

$$\log y = rt + \frac{a}{2\pi}\sin(2\pi t) + C$$

so

$$y = e^{rt + \frac{a}{2\pi}\sin(2\pi t) + C} = e^C e^{rt + \frac{a}{2\pi}\sin(2\pi t)}\,.$$

Noting that $y(t) = e^C$ we see that the solution is

$$y = y_0 \cdot e^{rt + \frac{a}{2\pi}\sin(2\pi t)}\,.$$

Finding a closed-form solution for the second equation would be more challenging.