Math 100C – SOLUTIONS TO WORKSHEET 3
THE DERIVATIVE

1. Three views of the derivative

(1) Let $f(x) = x^2$, and let $a = 2$. Then $(2, 4)$ is a point on the graph of $y = f(x)$.

(a) Let (x, x^2) be another point on the graph, close to $(2, 4)$. What is the slope of the line connecting the two? What is the limit of the slopes as $x \to 2$?

Solution: The slope of the line connecting two points is $\frac{\Delta y}{\Delta x}$, here $\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x + 2$, which tends to $\boxed{4}$ as $x \to 2$.

(b) Let h be a small quantity. What is the asymptotic behaviour of $f(2 + h)$ as $h \to 0$? What about $f(2 + h) - f(2)$?

Solution: $f(2 + h) = (2 + h)^2 = 4 + 4h + h^2 \sim 4 = f(2)$ as $h \to 0$ but then $f(2 + h) - f(2) = 4h + h^2 \sim \boxed{4}h$ as $h \to 0$.

(c) What is $\lim_{h\to 0} \frac{(2+h)^2-2^2}{h}$?

Solution: $\frac{(2+h)^2-2^2}{h} = \frac{4h+h^2}{h} = 4 + h \xrightarrow[h\to 0]{} \boxed{4}$

(d) What is the equation of the line tangent to the graph of $y = f(x)$ at $(2, 4)$?

Solution: We need a line of slope 4 through the point $(2, 4)$ so its equation is $y = 4(x-2)+4$.

(2) An analysis of market conditions indicate's your cousin's firm will generate a profit of $P(x) = 10x(7 - x) - 3x - 5$ if you produce x units of product. The firm is currently producing $x = 2$ units per month. Would you advise your cousin to increase to decrease production?

Solution: We have $P(2) = 10 \cdot 2 \cdot 5 - 3 \cdot 2 - 5 = 89$. If we marginally increase production by h units we have

$$\begin{aligned} P(2 + h) &= 10(2 + h)(7 - (2 + h)) - 3(2 + h) - 5 \\ &= 10(2 + h)(5 - h) - 11 - 3h \\ &= 100 + 30h - 10h^2 - 11 - 3h \\ &= 89 + 27h - 10h^2 \approx 89 + 27h \end{aligned}$$

to first order in h. We conclude that increasing production by h units will increase profits by about $27h$ – and in particular production should be increased.

Solution: Once we know about the derivative, we can write $P(x) = 67x - 10x^2 - 5$ so $P'(x) = 67 - 20x$ so $P'(2) = 27 > 0$ and the function is increasing about 0.

2. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ or $f(a + h) \approx f(a) + f'(a)h$

(3) Find $f'(a)$ if

(a) $f(x) = x^2$, $a = 3$.

Solution: $\lim_{h\to 0} \frac{(3+h)^2-(3)^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6 + h) = 6$.

Solution: $(3 + h)^2 = 3 + 6h + h^2 \approx 3 + 6h$ to second order so $f'(3) = 6$.

(b) $f(x) = \frac{1}{x}$, any a.

Solution: $\lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right) = \lim_{h\to 0} \frac{-h}{h \cdot a(a+h)} = -\lim_{h\to 0} \frac{1}{a(a+h)} = -\frac{1}{a^2}$.

Solution: $\frac{1}{a+h} - \frac{1}{a} = \frac{a}{a(a+h)} - \frac{a+h}{a(a+h)} = -\frac{h}{a(a+h)} \sim -\frac{h}{a^2}$ so $f'(a) = -\frac{1}{a^2}$.

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(c) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).

Solution: We have

$$\begin{aligned}\frac{(a+h)^3 - 2(a+h) - a^3 + 2a}{h} &= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h - a^3 + 2a}{h} \\ &= \frac{3a^2h + 3ah^2 + h^3 - 2h}{h} \\ &= 3a^2 - 2 + 3ah + h^2 \xrightarrow[h\to 0]{} 3a^2 - 2\,.\end{aligned}$$

Solution: We have

$$\begin{aligned}(a+h)^3 - 2(a+h) &= a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h \\ &= (a^3 - 2a) + (3a^2 - 2)h + 3ah^2 + h^3 \\ &\approx (a^3 - 2a) + (3a^2 - 2)h\end{aligned}$$

so the derivative is $3a^2 - 2$.

(4) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h) - \cos 5}{h}$, $\lim_{x\to 0} \frac{\sin x}{x}$

Solution: These are the derivative of $f(x) = \cos x$ at the point $a = 5$ and of $g(x) = \sin x$ at the point $a = 0$.

(5) (Final, 2015, variant – gluing derivatives) Is the function

$$f(x) = \begin{cases} x^2 & x \le 0 \\ x^2 \cos\frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?

Solution: We have $f(0) = 0$, so we'd have $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$ provided the limit exists, and since we have different expresions for $f(x)$ on both sides of 0 we compute the limit as two one-sided limits. On the left we have

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0\,.$$

Alternatively, we could recognize the limit as giving the derivative of $f(x) = x^2$ at $x = 0$. Using differentiation rules (to be covered later in the course) we know that $\left[\frac{d}{dx}x^2\right]_{x=0} = [2x]_{x=0} = 0$ and it would again follow that $\lim_{x\to 0^-} \frac{f(x)}{x} = 0$.

On the right we have

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2 \cos\frac{1}{x}}{x} = \lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = 0$$

since $x \to 0$ while $\cos\left(\frac{1}{x}\right)$ is bounded. Thus the function is differentiable and its derivative is zero.

3. The tangent line

(6) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

Solution: $f'(x) = \frac{1}{2\sqrt{x}}$, so the slope of the line is $f'(4) = \frac{1}{4}$, and the equation for the line line itself is $y - 2 = \frac{1}{4}(x - 4)$ or $y = \frac{1}{4}(x - 4) + 2$ or $y = \frac{1}{4}x + 1$.

(7) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

Solution: The line has slope 4 and meets the curve at $(1, 6)$. The last two functions don't evaluate to 6 at 1.We differentiate the first three.

$$\begin{aligned}\frac{d}{dx}|_{x=1} \left(x^3 + 2x^2 + 3x\right) &= \left(3x^2 + 4x + 3\right)|_{x=1} = 10 \\ \frac{d}{dx}|_{x=1} \left(x^2 + 3x + 2\right) &= (2x + 3)\,|_{x=1} = 5 \\ \frac{d}{dx}|_{x=1} \left(2\sqrt{x+3} + 2\right) &= \left(\frac{2}{2\sqrt{x+3}}\right)|_{x=1} = \frac{1}{2}\,.\end{aligned}$$

The answer is "none of the above".

(8) Find the lines of slope 3 tangent the curve $y = x^3 + 4x^2 - 8x + 3$.

Solution: $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 8x - 8$, so the line tangent at (x, y) has slope 3 iff $3x^2 + 8x - 8 = 3$, that is iff $3(x^2 - 1) + 8(x - 1) = 0$. We can factor this as $(x - 1)(3x + 11) = 0$ so the x-coordinates of the points of tangency are $1, -\frac{11}{3}$ and the lines are:

$$y = 3(x - 1)$$
$$y = 3(x + \frac{11}{3}) + \left(\left(\frac{11}{3}\right)^3 + 4\left(\frac{11}{3}\right)^2 - 8\left(\frac{11}{3}\right) + 3\right).$$

(9) The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

Solution: At the point (x, y) the curve has slope $\frac{dy}{dx} = 3x^2 + 2$, so the curve has slope 5 at the points where $x = \pm 1$, that is the points $(-1, -3)$ and $(1, 3)$. The line needs to meet the curve at the point, so there are two solutions:

$$y = 5x + 2 \qquad \text{(tangent at } (-1, -3))$$
$$y = 5x - 2 \qquad \text{(tangent at } (1, 3))$$

4. Linear approximation

Definition. $f(a + h) \approx f(a) + f'(a)h$

(10) Estimate

(a) $\sqrt{1.2}$

Solution: Let $f(x) = \sqrt{x}$ so that $f'(x) = \frac{1}{2\sqrt{x}}$. Then $f(1) = 1$ and $f'(1) = \frac{1}{2}$ so $f(1.2) \approx f(1) + f'(1) \cdot 0.2 = 1 + \frac{1}{2} \cdot 0.2 = 1.1$.

Better: $f(1.21) = 1.1$ and $f'(1.21) = \frac{1}{2.2}$ so $f(1.2) = f(1.21 - 0.01) \approx 1.1 - 0.01 \cdot \frac{1}{2.2} \approx 1.09545$.

(b) (Final, 2015) $\sqrt{8}$

Solution: Using the same f we have $f(9 - 1) \approx f(9) + f'(9) \cdot (-1) = 3 - \frac{1}{6} = 2\frac{5}{6}$.

(c) (Final, 2016) $(26)^{1/3}$

Solution: Let $f(x) = x^{1/3}$ so that $f'(x) = \frac{1}{3}x^{-2/3}$. Then $f(27) = 3$ and $f'(27) = \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27}$ so

$$f(26) = f(27 - 1) \approx f(27) + (-1) \cdot f'(27) = 3 - \frac{1}{27} = 2\frac{26}{27}.$$