11. Multivariable optimization (1/12/2022)

Goals.
(1) Critial points in 2d
(2) Multivariable optimization
(3) Constrained optimization

Last Time. functions of two (or more) variables.

① Graph $z = f(x,y)$ is a surface in 3d space.
(need to find points there, sometimes **project** onto 2d page)

② study change in f via **partial** derivatives

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

change x, keep y constant — change y, keep x constant.

Observation: If $f(x,y)$ has local max/min at (x_0, y_0) then $f(x, y_0)$ as a function of x has local max at x_0
so (1 var calc) $\boxed{\frac{\partial f}{\partial x}(x_0, y_0) = 0}$, similarly $\boxed{\frac{\partial f}{\partial y}(x_0, y_0) = 0}$

Call these the **critical points** of f.

Math 100C – WORKSHEET 11
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

(1) How many critical points does $f(x, y) = x^2 - x^4 + y^2$ have?

$\frac{\partial f}{\partial x} = 2x - 4x^3 = 2x(1-2x^2)$; $\frac{\partial f}{\partial y} = 2y$

So (x,y) is a critical point if $\begin{cases} 2x(1-2x^2) = 0 \\ 2y = 0 \end{cases}$

So $y=0$, $x \in \{0, \pm\frac{1}{\sqrt{2}}\}$.

So points are $\{(0,0), (\frac{1}{\sqrt{2}}, 0), (-\frac{1}{\sqrt{2}}, 0)\}$

Date: 1/12/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(2) Find the critical points of $f(x, y) = x^2 - x^4 + xy + y^2$.

$\frac{\partial f}{\partial x} = 2x - 4x^3 + y$; $\frac{\partial f}{\partial y} = x + 2y$

Need to solve $\begin{cases} 2x - 4x^3 + y = 0 \\ x + 2y = 0 \end{cases}$ if (x, y) is a solution

then $2(2x - 4x^3 + y) - (x + 2y) = 0$

so $3x - 8x^3 = 0$ so $3x\left(1 - \frac{8}{3}x^2\right) = 0$

so $x \in \left\{0, \sqrt{\frac{3}{8}}, -\sqrt{\frac{3}{8}}\right\}$ also $y = -\frac{1}{2}x$

so the critical points are $\left\{(0,0), \left(\sqrt{\frac{3}{8}}, -\frac{1}{2}\sqrt{\frac{3}{8}}\right), \left(-\sqrt{\frac{3}{8}}, \frac{1}{2}\sqrt{\frac{3}{8}}\right)\right\}$

(3) (MATH 105 Final, 2013) Find the critical points of $f(x, y) = xye^{-2x-y}$.

$\cancel{\frac{\partial f}{\partial y} = xe^{-2x-y} - xy}$

(5) Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.

$\frac{\partial f}{\partial x} = 7e^{-x-y} - (7x+3y+2y^2)e^{-x-y} = (7 - 7x - 3y - 2y^2)e^{-x-y}$

$\frac{\partial f}{\partial y} = (4y+3 - 7x - 3y - 2y^2)e^{-x-y}$

so critical points where $\begin{cases} 7-(7x+3y+2y^2)=0 \\ (4y+3)-(7x+3y+2y^2)=0 \end{cases}$ $e^{-x-y} \neq 0$ for all x, y

so $7 = 7x+3y+2y^2 = 4y+3$ so $\boxed{y=1}$

so $7 = 7x+5$ so $\boxed{x = \frac{2}{7}}$

only critical point at $(\frac{2}{7}, 1)$

2. Optimization

(6) Find the maximum of $(7x+3y+2y^2)e^{-x-y}$ for $x \geq 0$, $y \geq 0$,

If x or $y \to \infty$, the decaying exponential beats the polynomial ($f(x,y) \xrightarrow[x\to\infty \text{ or } y\to\infty]{} 0$) so maximum is either in interior (at critical pt) or on boundary $x=0$ or on boundary $y=0$

In the interior, one critical pt $(\frac{2}{7}, 1)$, $f(\frac{2}{7}, 1) = 7e^{-1\frac{2}{7}}$.

(4)

(a) Let $f(x,y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

$\frac{\partial f}{\partial x} = 8x$ vanishes at $x=0$, $\frac{\partial f}{\partial y} = 16y$ vanishe at $y=0$

Only critical point is $(0,0)$

This is the global minimum sice $f(x,y) \geq f(0,0)$ for all x,y $(4x^2 + 8y^2 \geq 0^2 + 0^2)$

(b) (MATH 105 Final, 2017) Let $f(x,y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x,y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

(7) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C = 3t^2 + \sqrt{t} \cdot q$. At price p the company can sell $q = \frac{t-p}{3}$ widgets.

(a) Write an expression for the profit function $f(q, t)$.

(b) How many widgets of what quality should the company make to maximize profits?

On $x=0$, $f(0,y) = (3y + 2y^2)e^{-y}$

$$\frac{\partial f}{\partial y}(0,y) = \left(3 + y - 2y^2\right)e^{-y}$$

Vanishes at

$$\frac{-1 \pm \sqrt{1^2+24}}{-2\cdot 2} = \frac{1 \pm 5}{4} = \frac{3}{2}, -1$$

On $[0,\infty)$, $f(0,0) = 0$, $f(0,\tfrac{3}{2}) = \left(3\cdot\frac{3}{2} + 2\cdot\frac{9}{4}\right)e^{-3/2}$

$\lim\limits_{y\to\infty} f(0,y) = 0$ ← e^{-y} beats $3y+2y^2$ $\quad = 9e^{-3/2}$

so maximum is $9e^{-3/2}$ at $(0, \tfrac{3}{2})$

On $y=0$, $f(x,0) = 7xe^{-x}$, $\frac{\partial f}{\partial x}(x,0) = (7-7x)e^{-x}$

has critical points $x=1$ where $f(1,0) = \frac{7}{e}$ $\quad = 7(1-x)e^{-x}$

of trace $x \mapsto f(x,0)$ $\quad f(0,0)=0$, $\lim\limits_{x\to\infty} f(x,0) = 0$

same as before

maximum here is $\frac{7}{e}$, attained at $(1,0)$

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so $\max\left\{ f(x,y) \,\middle|\, \begin{matrix} x\geq 0 \\ y \geq 0\end{matrix} \right\}$ is the largest of

$$\frac{7}{e^{9/7}}, \quad \frac{9}{e^{3/2}}, \quad \frac{7}{e}$$

So $\frac{7}{e}$ is largest.

Multivariable optimization

1. find critical pts +singular (if any) in domain, evaluate ~~fcn.~~ function at points
2. Optimize on ~~bdry~~ boundary **(see next)**
3. Take largest / smallest value as needed

Problem: (e.g. optimize on boundary)

Find max/min of $f(x,y)$ ← "objective function"

subject to $G(x,y)=0$ ← "constraint"

(Called "constrained optimization")

One method: along $G(x,y)=0$ solve for $y=y(x)$

Then on $G(x,y)=0$ have $f(x,y)=f(x,y(x))$

now do 1-var calculus

Method of Lagrange multipliers:

Add variable λ, replace "critical pts" with system

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = \lambda \frac{\partial G}{\partial x} \\ \frac{\partial f}{\partial y}(x,y) = \lambda \frac{\partial G}{\partial y} \end{cases} \qquad G(x,y)=0$$

(8) (MATH 105 final, 2017) Use the method of Lagrange Multipliers to find the maximum value of the utility function $U = f(x,y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x,y) = 50x + 100y - 500,000 = 0$, where $x \geq 0$ and $y \geq 0$.

$f(0,y) = f(x,0) = 0$ So maximum is in interior where Lagrange multipliers apply. Get:

$$\begin{cases} (1) & 4x^{-\frac{3}{4}}y^{3/4} = \lambda \cdot 50 \\ (2) & 12x^{\frac{1}{4}}y^{-\frac{1}{4}} = \lambda \cdot 100 \\ (3) & 50x + 100y = 500{,}000 \end{cases}$$

divide $\frac{\text{Eqn } (2)}{\text{Eqn } (1)}$

Get: $$\begin{cases} 3xy^{-1} = 2 \iff 3x = 2y \\ x + 2y = 10{,}000 \end{cases}$$

So $4x = x + 3x = x + 2y = 10{,}000$ so $x = 2{,}500$

So $y = \frac{3x}{2} = \frac{7{,}500}{2} = 3{,}750$

Max: $f(2{,}500, 3750) =$

4. Combination problems

(10) Find the maximum and minimum values of $f(x,y) = -x^2 + 8y$ in the disc $R = \{x^2 + y^2 \leq 25\}$.

Since R is closed, bounded f will have max/min (f is ctsl)

~~If~~ in the interior, max/min will be at critical pts

$\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = 8 \leftarrow$ ~~never~~ never vanishes, so no critical pts

So max/min occur on the boundary $x^2+y^2=25$

Apply Lagrange multipliers: max/min occur where

$$\begin{cases} -2x = \lambda \cdot 2x \\ 8 = \lambda \cdot 2y \\ x^2+y^2=25 \end{cases}$$

1st equation holds if

(1) $\lambda = -1$

(2) $x = 0$

If $\lambda = -1$ 2nd equation gives $y = -4$, so ~~x=±3~~ $x = \pm\sqrt{25-4^2} = \pm 3$

If $x=0$, $y = \pm 5$. Need to check: $(\pm 3, -4)$

$(0, \pm 5)$

$f(\pm 3, -4) = -9 - 48 = -41$

$f(0,5) = 40, \quad f(0,-5) = -40$

$\Rightarrow$ max $= 40$

min $= -41$