Math 100C, lecture 2, 22/9/22

Goals

(1) Limits

(2) Asymptotes

Last time: Asymptotics as $x \to \infty$, $x^2 + x \sim x^2$

$e^x + x^3 \sim e^x$

$e^{x^2} + e^x \sim e^{x^2}$

$\vdots$

as $x \to 0$ $\quad x + x^2 \sim x$

as $x \to 0$ $\quad 1 + x \sim 1$

More complicated expressions: Assembled from pieces ("parse trees")

(call $-x$ "minus x")

WS 1

LIMITS AND ASYMPTOTES

(1) Review of asymptotics: analyze the expression $\frac{e^x + A\sin x}{e^x - x^2}$ as $x \to \infty$, $x \to 0$, $x \to -\infty$.

As $x \to \infty$ $A\sin x$ bounded, dominated by e^x
e^x dominates x^2, so $e^x + A\sin x \sim e^x$, $e^x - x^2 \sim e^x$ $\Rightarrow \frac{e^x + A\sin x}{e^x - x^2} \sim 1$

As $x \to 0$, $e^x + A\sin x \to 1$, $e^x - x^2 \to 1$ so $\frac{e^x + A\sin x}{e^x - x^2} \sim 1$

As $x \to -\infty$, $e^x - x^2 \sim -x^2$ but $e^x + A\sin x$ has no clean asymptotics (still bdd by $|A|+1$)

1. Limits

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x \to 5}(x^3 - x) = 5^3 - 5 = 120$

Fact: If $f(x)$ is defined by expression, $f(a)$ makes sense then

$$\lim_{x \to a} f(x) = f(a)$$

Date: 22/9/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

① Limits

The limit of $f(x)$ as $x \to a$ is the value $f(x)$ 'would like to have' at a, i.e the value it approaches.

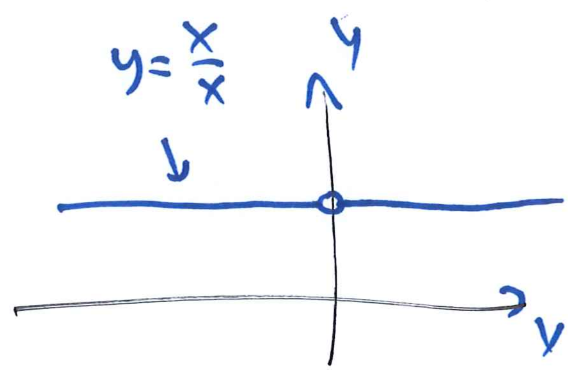
Asymptotics: how we approach the limit

E.g's $\lim_{x \to \infty} e^{-x} = 0$ $\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to 0} x = 0$, $\lim_{x \to 0} x^2 = 0$

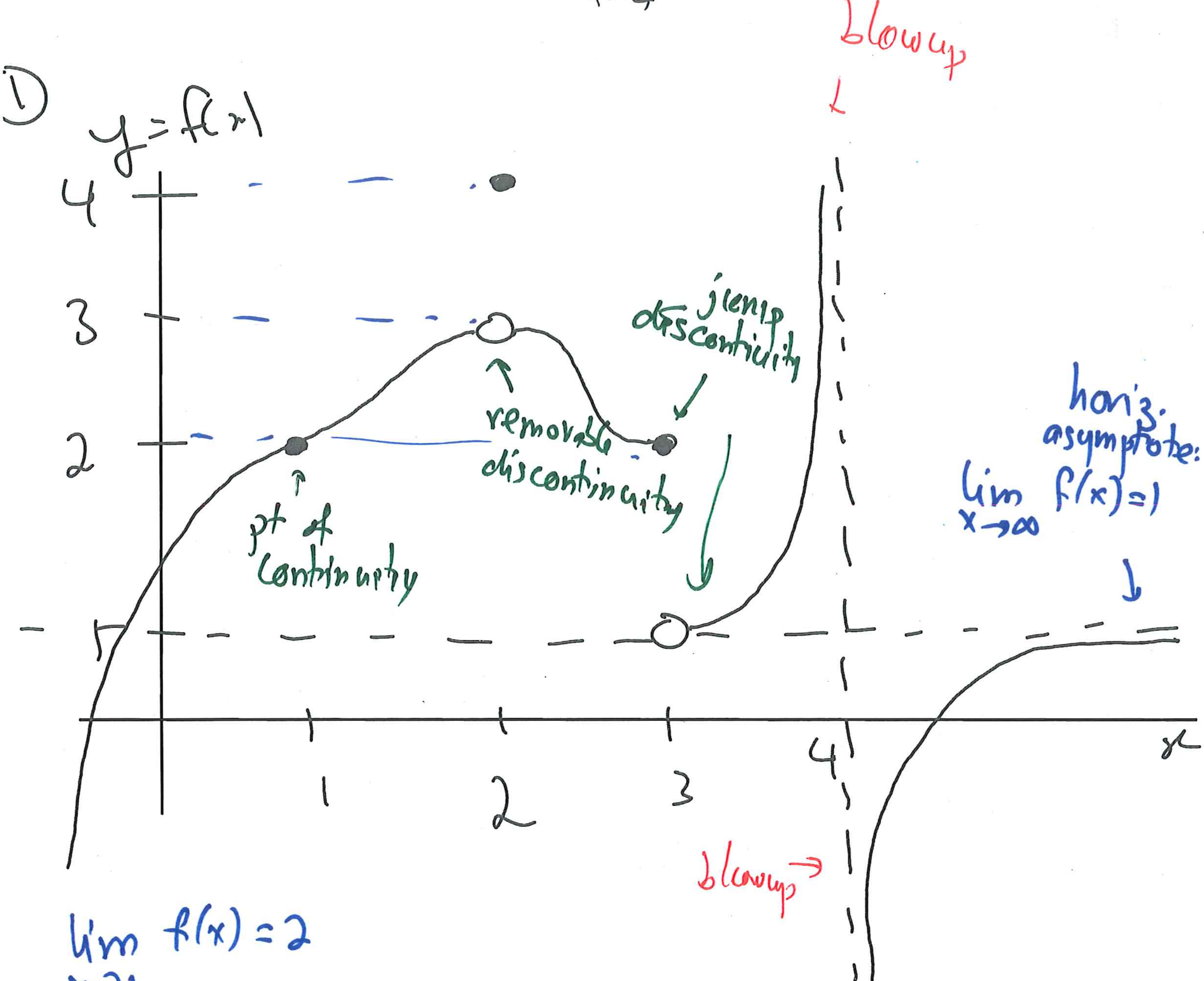
W.S (2) (8)

Example: $\lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$



$\lim_{x\to 4^-} f(x)$ DNE but $\lim_{x\to 4^-} f(x) = +\infty$ (in the extended sense)

$\lim_{x\to 4^+} f(x)$ DNE but $\lim_{x\to 4^+} f(x) = -\infty$



$\lim_{x\to 1} f(x) = 2$

$\lim_{x\to 2} f(x) = 3$ but $f(2) = 4$

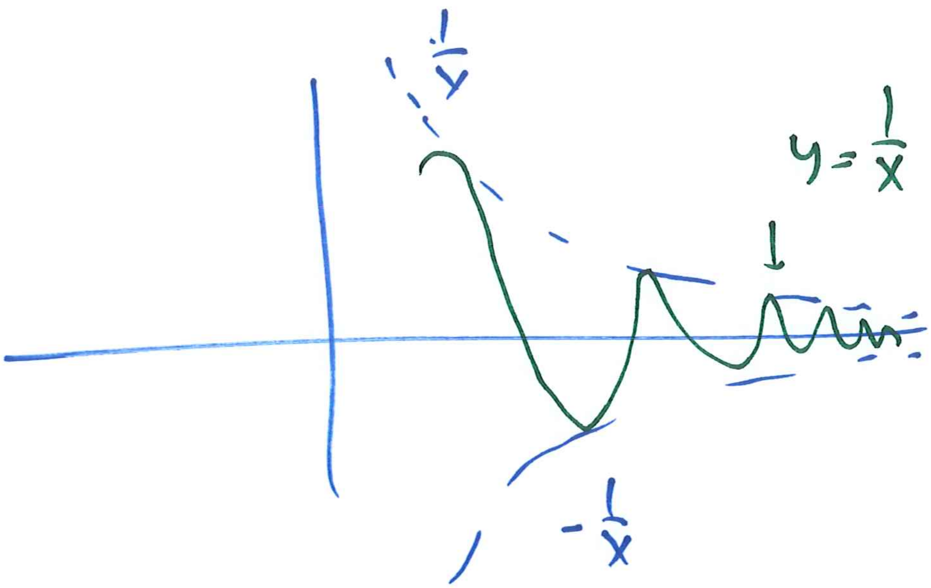
$\lim_{x\to 3} f(x)$ DNE but $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to 3^+} f(x) = 1$

"does not exist"

If as $x \to +\infty$ (or as $x \to -\infty$)

$\lim_{x \to \infty} f(x) = L$ we say " $y = L$ is a horizontal asymptote of f as $x \to \infty$)

(or $-\infty$)



$\lim_{x \to \infty} \frac{\sin x}{x} = 0$

✓ have horiz. asymptote $y = 0$

If ~~as~~ $x \to a$ (or $x \to a^+$, $x \to a^-$)

the function values $f(x)$ grow to ∞ (or to $-\infty$)

we say ~~the~~ (1) the function has a vertical asymptote at a

(2) the function "blows up" at a (at $x=a$)

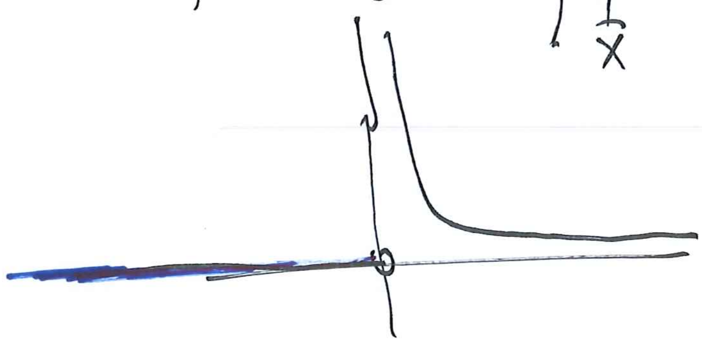
Example: $f(x) = \frac{1}{x}$ blows up at $x=0$

Example: $g(x) = \begin{cases} 0 & x<0 \\ \frac{1}{x} & x>0 \end{cases}$

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 0 = 0$

if $x<0$, $g(x)=0$

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty$



Example: Bank of Canada interest rates

$I(t) = \begin{cases} 3.5\% & t > 16/9/2022 \\ 2.75\% & 22/7/2022 < t < 16/9/2022 \\ 1.5\% & 1/6/2022 < t < 22/7/2022 \end{cases}$

in each interval defined by formula,
but at breakpoints needs separate analysis

wrong: $\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \frac{1}{x+4} = \lim_{x\to 3} \frac{1}{x+4} = \frac{1}{7}$

↑ number function

(3) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

note. $x^2+x-12 = (x-3)(x+4)$

so $\frac{x-3}{x^2+x-12} = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4} \xrightarrow[x\to 3]{} \frac{1}{7}$

(b) What about $\lim_{x\to -4} f(x)$?

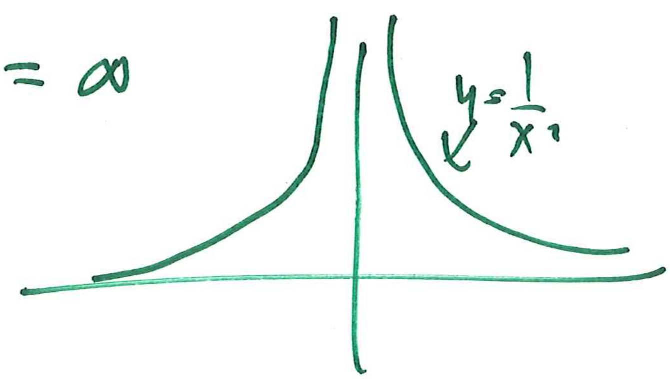
blowup. If $x < -4$, $\frac{1}{x+4} < 0$ so $\lim_{x\to -4^-} \frac{1}{x+4} = -\infty$

if $x > -4$, $\frac{1}{x+4} > 0$, so $\lim_{x\to -4^+} \frac{x-3}{x^2+x-12} = +\infty$

(to tell if f blows up to $+\infty$ or to $-\infty$ examine its sign) so $\lim_{x\to -4} f(x)$ DNE

---

but: $\lim_{x\to -4} \frac{1}{(x+4)^2} = \infty$



as $x \to \infty$

(4) Evaluate

(a) $\lim_{x\to\infty} \frac{e^x + A\sin x}{e^x - x^2}$! $\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{e^x}{e^x} \sim 1$

so $\lim_{x\to\infty} \frac{e^x + A\sin x}{e^x - x^2} = 1$

(b) $\lim_{x\to 0} \frac{e^x + A\sin x}{e^x - x^2} = \frac{e^0 + A\sin 0}{e^0 - 0^2} = 1$

defined by formula, makes sense at $x=0$

(c) $\lim_{x\to -\infty} \frac{e^x + A\sin x}{e^x - x^2}$ as $x \to -\infty$, $e^x - x^2 \sim -x^2$

$e^x + A\sin x$ is bounded

$\frac{1}{x^2} \to 0$ so $\lim_{x\to-\infty} \frac{e^x + A\sin x}{e^x - x^2} = 0$
$x \to -\infty$

(5) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2 - 1} = \frac{3}{15} = \frac{1}{5}$

expression makes sense at $x=2$

(b) (Final, 2014) $\lim_{x\to -3^+} \frac{x+2}{x+3}$. as $x \to -3$, $x+2 \to -1$

so $\frac{x+2}{x+3} \sim \frac{-1}{x+3}$ as $x \to -3$ if $x > -3$, this is < 0

so $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$

(c) $\lim_{x\to 1}\frac{e^x(x-1)}{x^2+x-2}$

(d) $\lim_{x\to -2^-}\frac{e^x(x-1)}{x^2+x-2}$

(e) $\lim_{x\to 1}\frac{1}{(x-1)^2}$

(f) $\lim_{x\to 4}\frac{\sin x}{|x-4|}$: as $x\to 4$ $\frac{\sin x}{|x-4|} \sim \frac{\sin 4}{|x-4|}$

So have blowup, $\pi < 4 < 2\pi$ so $\sin 4 < 0$, $|x-4| > 0$

so $\lim_{x\to 4}\frac{\sin x}{|x-4|} = -\infty$

(g) $\lim_{x\to\frac{\pi}{2}^+}\tan x$, $\lim_{x\to\frac{\pi}{2}^-}\tan x$.

2. Limits at infinity

(6) Evaluate

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3}$

as $x \to \infty$ $\quad \frac{x^2+1}{x-3} \sim \frac{x^2}{x} \sim x \underset{x\to\infty}{\to} \infty$

(b) (Final, 2015) $\lim_{x\to-\infty} \frac{x+1}{x^2+2x-8}$