Math 100 – SOLUTIONS TO WORKSHEET 12
EXPONENTIAL GROWTH AND DECAY

1. More Related Rates

(1) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

Solution: The water fills a conical volume inside the drain. Suppose that at time t the height of the water is $h(t)$ and the radius at the surface of the water is $r(t)$. Then by similar triangles

$$\frac{r(t)}{h(t)} = \frac{1}{6}\,.$$

We therefore have $r(t) = \frac{h(t)}{6}$. The volume of the water is therefore

$$V(t) = \frac{1}{3}\pi r^2 h = \frac{\pi}{108}h^3(t)\,.$$

Differentiating we find

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{36}h^2(t)\frac{\mathrm{d}h}{\mathrm{d}t}\,.$$

In particular, if $\frac{\mathrm{d}V}{\mathrm{d}t} = 5\text{m}^3/\text{min}$ and $h = 5\text{m}$ then

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{36\cdot 5}{\pi\cdot 5^2} = \frac{36}{5\pi}\frac{\text{m}}{\text{min}}\,.$$

(b) The drain is unclogged and water begins to clear at the rate of $\frac{\pi}{4}\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Solution: We are now given $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\pi}{4}\frac{\text{m}^3}{\text{min}}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = -1\frac{\text{m}}{\text{min}}$. Then

$$h(t) = \sqrt{\frac{36\frac{\mathrm{d}V}{\mathrm{d}t}}{\pi\frac{\mathrm{d}h}{\mathrm{d}t}}} = \sqrt{\frac{-36\pi}{4\pi(-1)}} = \sqrt{9} = 3\text{m}\,.$$

(2) Two ships are travelling near an island. The first is located 20km due west of it, The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing if:

(a) The first ship is moving due north at 5km/h.

Solution: Suppose the ships are A, B at positions $(x_A, y_A), (x_B, y_B)$. Then the distance between them satisfies

$$D^2 = (x_A - x_B)^2 + (y_A - y_B)^2\,.$$

It follows that

$$2D\cdot\frac{\mathrm{d}D}{\mathrm{d}t} = 2(x_A - x_B)\left(\frac{\mathrm{d}x_A}{\mathrm{d}t} - \frac{\mathrm{d}x_B}{\mathrm{d}t}\right) + 2(y_A - y_B)\left(\frac{\mathrm{d}y_A}{\mathrm{d}t} - \frac{\mathrm{d}y_B}{\mathrm{d}t}\right)\,.$$

At the given time $\frac{\mathrm{d}x_A}{\mathrm{d}t} = \frac{\mathrm{d}x_B}{\mathrm{d}t} = 0$ (the ships are moving north/south), $y_A = 0$, $y_B = -15\text{km}$, $\frac{\mathrm{d}y_A}{\mathrm{d}t} = 5\text{km/h}$ and $\frac{\mathrm{d}y_B}{\mathrm{d}t} = -7\text{km/h}$. Finally at the given time $D = \sqrt{15^2 + 20^2}\text{km} = 25\text{km}$. At the given time we therefore have

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{(0 - (-15))}{25}(5 - (-7)) = \frac{15\cdot 12}{25} = \boxed{7.2\text{km/h}}\,.$$

(b) The same setting, but now the first ship is moving toward the island.

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Solution: Now $\frac{\mathrm{d}x_A}{\mathrm{d}t} = 5\text{km/h}$ but $\frac{\mathrm{d}x_B}{\mathrm{d}t} = 0$, $x_A = -20\text{km}$, $x_B = y_A = 0$, $y_B = -15\text{km}$, $\frac{\mathrm{d}y_A}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}y_B}{\mathrm{d}t} = -7\text{km/h}$. Again $D = 25\text{km}$. so

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}t} &= \frac{1}{25}\left[(-20-0)(5-0) + (0-(-15))(0-(-7)\right] \\ &= \frac{1}{25}\left[-100+105\right] = \boxed{\frac{1}{5}\text{km/h}}.\end{aligned}$$

2. Exponential growth and decay

(3) Suppose[1] that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.

(a) At what time will there be 1000 opossums in BC? 10,000 opossums?
Solution: After t years there are $2 \cdot 3^t$ opossums, so there will be 1000 opossums after $\frac{\log 500}{\log 3} \approx 5.7$ years and $10,000$ opossums after $\frac{\log 5000}{\log 3} \approx 7.8$ years.

(b) Write a differential equation expressing the growth of the opossum population with time.
Solution: Since $\frac{\mathrm{d}}{\mathrm{d}t}3^t = \log 3 \cdot 3^t$ the differential equation is $\frac{\mathrm{d}N}{\mathrm{d}t} = \log 3 \cdot N$.

(4) A radioactive sample decays according to the law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km\,.$$

(a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?

(b) A 100-gram sample is left unattended for three days. How much of it remains?
Solution: If two halvings happened in 10 hours, the half-life is 5 hours. Accordingly after three days we will have $\frac{72}{5}$ half-lives, so the remaining mass will be

$$100 \cdot 2^{-\frac{72}{5}} = 100 \cdot \exp\left(-\frac{72}{5}\log 2\right) \approx 0.005\text{g}\,.$$

(5) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.
Solution: After 6 hours we've had 1.5 doublings, so the original size is $\frac{2000}{2^{1.5}} = 1000\frac{\sqrt{2}}{2} \approx 707$.

3. Newton's Law of Cooling

Fact. *When a body of temperature T_0 is placed in an environment of temperature T_{env} the temperature difference $T(t) - T_{env}$ between the body and the environment decays exponentially. In other words, there is a (negative) constant k such that*

$$T' = k(T - T_{env}) \qquad\qquad T(t) - T_{env} = (T_0 - T_{env})e^{kt}\,.$$

- *key idea*: change variables to the *temperature difference*. Let $y = T - T_{\text{env}}$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} - 0 = ky$$

Corollary. $\lim_{t\to\infty} y(t) = 0$ *so* $\lim_{t\to\infty} T(y) = T_{env}$.

(6) (Final, 2010) When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.
Solution: Let $T(t)$ be the temperature of the apple t minutes after it was taken from the refrigerator, and let $y(t) = T(t) - 19$ be the temperature difference. Newton's law of cooling provides that $y(t)$ decays exponentially at constant rate. We are given that $y(0) = -16°C$ and that $y(30) = -8°C$ so the temperature difference was halved after 30 minutes. By 90 minutes there would be two further halvings, so $y(90) = -2°C$ and $T(90) = -2 + 19 = 17°C$.

[1] See http://linnet.geog.ubc.ca/efauna/Atlas/Atlas.aspx?sciname=Didelphis%20virginiana

(b) Determine the time when the temperature of the apple is $16°C$.
Solution: We are asked when the temperature difference will be $-3°C$. Since the temperature difference satisfies the law $y(t) = -16°C \cdot 2^{-t/30}$ we need to find t so that

$$-3 = -16 \cdot 2^{-t/30}$$

that is

$$2^{t/30} = \frac{16}{3}\,.$$

Taking logarithms of both sides we have

$$\frac{t}{30}\log 2 = \log 16 - \log 3$$

so that the apple reaches $16°C$ at time

$$t = 30 \cdot \frac{\log 16 - \log 2}{\log 2}\ \text{minutes.}$$

(c) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.
Solution: We are asked when the temperature difference will be $-3°C$. Since the temperature difference satisfies the law $y(t) = -16°C \cdot 2^{-t/30}$ we need to find t so that

$$-3 = -16 \cdot 2^{-t/30}$$

that is

$$2^{t/30} = \frac{16}{3}\,.$$

Taking logarithms of both sides we have

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so that the apple reaches $16°C$ at time

$$t = 30 \cdot \frac{\log 16 - \log 2}{\log 2}\ \text{minutes.}$$

The temperature of the apple at time T is $T(t) = 19°C - 16°C \cdot 2^{-t/30} = T = 19°C - 16°C \cdot e^{-\frac{\log 2}{30}t}$.We therefore have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\log 2}{30}\,(T - 19)\,.$$

(7) (Final, 2013) A bottle of soda pop at room temperature ($70°F$) is placed in the refrigerator where the temperature is $40°F$. After half han hour the bottle has cooled to $60°F$. When will it reach $50°F$?

Solution: Let $T(t)$ be the temperature of the soda t minutes after it was put in the fridge, and let $y(t) = T(t) - 40°F$ be the temperature difference. Then we are given that $y(0) = 30°F$ and that $y(30) = 20°F$. Newton's law of cooling provides that $y(t)$ decays exponentially at constant rate, so we conclude that $y(t) = 30°F \cdot \left(\frac{2}{3}\right)^{t/30}$. We are asked for the time when $T(t) = 50°F$, that is when $y(t) = 10°F$. That time therefore satisfies:

$$10 = 30 \cdot \left(\frac{2}{3}\right)^{t/30}\,,$$

that is

$$3 = \left(\frac{3}{2}\right)^{t/30}\,.$$

Taking logarithms we find

$$\log 3 = \frac{t}{30}\log\frac{3}{2}$$

so

$$t = 30\frac{\log 3}{\log 3 - \log 2}\ \text{minutes.}$$