Math 100 – SOLUTIONS TO WORKSHEET 11
INVERSE TRIG FUNCTIONS; RELATED RATES

1. Inverse trig functions

(1) Evaluation
  (a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.
  Solution: $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$. Also $\sin\left(\frac{31\pi}{11}\right)=\sin\left(\frac{31\pi}{11}-2\pi\right)=\sin\left(\frac{9\pi}{11}\right)=\sin\left(\pi-\frac{9\pi}{11}\right)=\sin\left(\frac{2\pi}{11}\right)$ and $\frac{2\pi}{11}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)=\frac{2\pi}{11}$.
  (b) (Final 2015) Simplify $\sin(\arctan 4)$
  Solution: Consider the right-angled triangle with sides 4, 1 and hypotenuse $\sqrt{1+4^2}=\sqrt{17}$. Let θ be the angle opposite the side of length 4. Then $\tan\theta=4$ and $\sin\theta=\frac{4}{\sqrt{17}}$ so $\sin(\arctan 4)=\sin\theta=\frac{4}{\sqrt{17}}$.
  (c) Find $\tan(\arccos(0.4))$
  Solution: Consider the right-angled triangle with sides 0.4, $\sqrt{1-0.4^2}$ and hypotenuse 1. Let θ be the angle between the side of length 0.4 and the hypotenuse. Then $\cos\theta=\frac{0.4}{1}=0.4$ and $\tan\theta=\frac{\sqrt{1-0.4^2}}{0.4}=\frac{\sqrt{0.84}}{0.4}=\sqrt{\frac{0.84}{0.16}}=\sqrt{5.25}$.

(2) Differentiation
  (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$
  Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x)=\frac{1}{\sqrt{1-x^2}}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))=\frac{2}{\sqrt{1-4x^2}}.$$

  Alternatively, let $\theta=\arcsin 2x$, so that $\sin\theta=2x$. Differentiating both sides we get

$$\cos\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x}=2$$

  so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{2}{\cos\theta}=\frac{2}{\sqrt{1-\sin^2\theta}}=\frac{2}{\sqrt{1-4x^2}}.$$

  (b) Find the line tangent to $y=\sqrt{1+(\arctan(x))^2}$ at the point where $x=1$.
  Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x)=\frac{1}{1+x^2}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+(\arctan(x))^2}=\frac{1}{2\sqrt{1+(\arctan(x))^2}}\cdot 2\arctan(x)\cdot\frac{1}{1+x^2}$$
$$=\frac{\arctan x}{(1+x^2)\sqrt{1+(\arctan(x))^2}}.$$

  Now $\arctan 1=\frac{\pi}{4}$ so the line is

$$y=\frac{\pi}{8\sqrt{1+\frac{\pi^2}{16}}}(x-1)+\sqrt{1+\frac{\pi^2}{16}}.$$

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(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?
Solution: From the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(e^{5x}\right) = \frac{1}{\sqrt{1-e^{10x}}}5e^{5x} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}\,.$$

The function y itself is defined when $-1 \le e^{5x} \le 1$, that is when $5x \le 0$, that is when $x \le 0$. The derivative is defined when $-1 < e^{10x} < 1$, that is when $x < 0$. The point is that since $\sin\theta$ has horizontal tangents at $\pm\frac{\pi}{2}$, $\arcsin x$ has vertical tangents at ± 1.
Solution: We can write the identity as $\sin y = e^{5x}$ and differentiate both sides to get $y'\cos y = 5e^{5x}$ so that

$$y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1-\sin^2 y}} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}\,.$$

2. Velocity and acceleration

(3) A particle's position is given by $f(t) = t + 6e^{-t/3}$.
(a) Find the velocity at time t, and specifically at $t = 3$.
Solution: The velocity is the derivative of the position, so $v(t) = \frac{\mathrm{d}f}{\mathrm{d}t} = 1 - 2e^{-t/3}$. In particular $v(3) = 1 - \frac{2}{e}$.
(b) When is the particle moving to the right? to the left?
Solution: The particular is moving to the right when $1 - 2e^{-t/3} > 0$ i.e. when $2e^{-t/3} < 1$ or when $e^{t/3} > 2$ that is when $t > 3\log 2$. It is moving to the left when $t < 3\log 2$.
(c) When is the particle accelerating? decelerating?
Solution: The acceleration is $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{2}{3}e^{-t/3}$. This is always positive. But "accelerating" means the acceleration is in the same direction as the velocity! So the particle is accelerating when $t > 3\log 2$ and deccelerating before that. **Warning**: the coincidence of the times in parts (b),(c) is not a general fact!

(4) (Final, 2016) An object is thrown straight up into the air at time $t = 0$ seconds. Its height in metres at time t seconds is given by $h(t) = s_0 + v_0 t - 5t^2$. In the first second the object rises by 5 metres. For how many seconds does the object rise before beginning to fall?
Solution: We are given that $h(1) - h(0) = 5$, in other words that $(s_0 + v_0 - 5) - s_0 = 5$ so that $v_0 = 10$. Now the velocity of the object is

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = v_0 - 10t = 10 - 10t$$

and this is positive as long as $t \le 1$s.

(5) A emergency breaking car can decelerate at $9\frac{\mathrm{m}}{\mathrm{s}^2}$. How fast can a car drive so that it can come to a stop within 50m?
Solution: Suppose the car begins with velocity v_0. Its velocity at time t is then $v(t) = v_0 - gt$ so the stopping time is $t = \frac{v_0}{g}$. Reversing time, the distance travelled during the deceleration is the same as the distance travelled while accelerating at acceleration g for time t. The breaking distance L therefore has the form $\frac{1}{2}gt^2 = \frac{v_0^2}{2g}$. The maximum speed is then

$$v_0 = \sqrt{2gL} = \sqrt{2\cdot 9\cdot 50} = 30\frac{\mathrm{m}}{\mathrm{s}} = 180\mathrm{km/h}.$$

3. Related Rates

(6) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $\left(1, \sqrt{3}\right)$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.
Solution: We differentiate along the curve with respect to time, finding

$$2y\frac{\mathrm{d}y}{\mathrm{d}t} = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

Plugging in $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$, $x = 1$, $y = \sqrt{3}$ we find: $2\sqrt{3} = 5\frac{\mathrm{d}x}{\mathrm{d}t}$ so at that time we have

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2\sqrt{3}}{5}.}$$

(7) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

Solution: The water fills a conical volume inside the drain. Suppose that at time t the height of the water is $h(t)$ and the radius at the surface of the water is $r(t)$. Then by similar triangles

$$\frac{r(t)}{h(t)} = \frac{1}{6}.$$

We therefore have $r(t) = \frac{h(t)}{6}$. The volume of the water is therefore

$$V(t) = \frac{1}{3}\pi r^2 h = \frac{\pi}{108} h^3(t).$$

Differentiating we find

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{36} h^2(t) \frac{\mathrm{d}h}{\mathrm{d}t}.$$

In particular, if $\frac{\mathrm{d}V}{\mathrm{d}t} = 5\text{m}^3/\text{min}$ and $h = 5\text{m}$ then

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{36 \cdot 5}{\pi \cdot 5^2} = \frac{36}{5\pi} \frac{\text{m}}{\text{min}}.$$

(b) The drain is unclogged and water begins to drain at the rate of $(5 + \frac{\pi}{4})\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of 1m/min?

Solution: We are now given $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\pi}{4}\frac{\text{m}^3}{\text{min}}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = -1\frac{\text{m}}{\text{min}}$. Then

$$h(t) = \sqrt{\frac{36\frac{\mathrm{d}V}{\mathrm{d}t}}{\pi\frac{\mathrm{d}h}{\mathrm{d}t}}} = \sqrt{\frac{-36\pi}{4\pi(-1)}} = \sqrt{9} = 3\text{m}.$$