Math 100 – SOLUTIONS TO WORKSHEET 4
CONTINUITY; THE IVT

1. Continuity

(1) Which of these functions are continuous everywhere? Why?

(a) $f(x) = \begin{cases} x & x < 0 \\ \cos x & x \geq 0 \end{cases}$

Solution: This is clearly continuous on $(-\infty, 0)$ and $(0, \infty)$ but $\lim_{x\to 0^-} f(x) = 0$ while $\lim_{x\to 0^+} f(x) = 1$.

(b) $g(x) = \begin{cases} x & x < 0 \\ \sin x & x \geq 0 \end{cases}$

Solution: This is clearly continuous on $(-\infty, 0)$ and $(0, \infty)$. Also, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0 = f(0)$ so the function is continuous everywhere.

(2) Let $f(x) = \frac{x^3-x^2}{x-1}$.

(a) Why is $f(x)$ discontinuous at $x = 1$?

Solution: The formula is undefined there.

(b) Find b such that $g(x) = \begin{cases} f(x) & x \neq 1 \\ b & x = 1 \end{cases}$ is continouous everywhere.

Solution: $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2(x-1)}{x-1} = \lim_{x\to 1} x^2 = 1$ so setting $b = 1$ gives the desired function.

(c) Find c, d such that $h(x) = \begin{cases} \sqrt{x} & 0 \leq x < 1 \\ c & x = 1 \\ d - x^2 & x > 1 \end{cases}$ is continuous.

Solution: The function is already continuous on $[0, 1)$ and $(1, \infty)$. We have $\lim_{x\to 1^-} h(x) = \lim_{x\to 1} \sqrt{x} = 1$ so we must have $c = 1$. We also need $1 = \lim_{x\to 1^+} h(x) = \lim_{x\to 1^+} \left(d - x^2\right) = d - 1$ so we need $d = 2$.

(d) (Final 2013) For which value of the constant c is $f(x) = \begin{cases} cx^2 + 3 & x \geq 1 \\ 2x^3 - c & x < 1 \end{cases}$ continuous on $(-\infty, \infty)$?

Solution: The function is already continuous on $(-\infty, 1)$ and $(1, \infty)$. We have $\lim_{x\to 1^-} f(x) = 2 \cdot 1^3 - c = 2 - c$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} c \cdot 1^2 + 3 = 3 + c = f(1)$ so the function will be continuous at $x = 1$ iff $2 - c = 3 + c$ that is if $\boxed{c = -\frac{1}{2}}$.

(3) Where are the following functions continuous?

Solution: $\left(-\sqrt{7}, \sqrt{7}\right)$, $(-\infty, +\infty)$, $(-\infty, -1) \cup (-1, \infty)$ respectively. In the last case, the denominator vanishes at $x = 1$ and the function actually blows up there ($2+\cos 1 \neq 0$). The logarithm is defined only for positive arguments, so the domain is $\{x \mid \sin x > 0\} = \bigcup_{k\in\mathbb{Z}} \left(2\pi k - \frac{\pi}{2}, 2\pi k + \frac{\pi}{2}\right)$

(4) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x\to 3} (xf(x) + g(x)) = 1$. Find $f(3)$.

Solution: Since f, g are continuous and applying the limit laws we have

$$\begin{aligned} 1 = \lim_{x\to 3} (xf(x) + g(x)) &= \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} f(x)\right) + \left(\lim_{x\to 3} g(x)\right) \\ &= 3f(3) + g(3) = 3f(3) + 2\,. \end{aligned}$$

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Solving for $f(3)$ we get

$$\boxed{f(3) = -\frac{1}{3}}.$$

2. The Intermediate Value Theorem

Theorem. *Let $f(x)$ be continuous for $a \le x \le b$. Then $f(x)$ takes every value between $f(a), f(b)$.*

(5) Show that $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.

Solution: f is continuous on $[0,1]$ (given by formula there). We have $f(0) = 1$, $f(1) = -2$. By the intermediate value theorem there is $x_0 \in (0,1)$ such that $f(x_0) = 0 \in (-2,1)$.

(6) (Final 2011) Let $y = f(x)$ be continuous with domain $[0,1]$ and range in $[3,5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

Solution: Consider the diffference $g(x) = f(x) - (2x+3)$. By arithmetic of limits this is a continuous function. We have $g(0) = f(0) - 3 \ge 3 - 3 = 0$ (since $f(0) \ge 3$). We have $g(1) = f(1) - 5 \le 5 - 5 = 0$. By the IVT $g(x)$ takes every value between $g(0)$ and $g(1)$, so there is x_0 such that $g(x_0) = 0$ and then $f(x_0) - (2x_0 + 3) = 0$ so $f(x_0) = 2x_0 + 3$ so the graphs intersect at the point $(x_0, 2x_0 + 3)$.

(7) (Final 2010) Two points on Earth are called *antipodal* if they are exactly opposite to each other. Show that, at any given moment, there are two antipodal points on the equator with exactly the same temperature.

Solution: Let $T(\theta)$ be the temperature at the point on the equator with longitude θ (say measured in radians), which we suppose continuous. Let $f(\theta) = T(\theta) - T(\theta + \pi)$ be the difference between the temperatures at a point on the equator and its antipode. Then f is continuous as well since it is the difference of continuous functions, and in terms of f we are looking for longitudes θ such that $f(\theta) = 0$. Now choose any point θ_0. If $f(\theta_0) = 0$ we are done. Otherwise $f(\theta_0)$ is either positive or negative, and we observe that $f(\theta_0 + \pi) = T(\theta_0 + \pi) - T(\theta_0 + 2\pi) = T(\theta_0 + \pi) - T(\theta_0) = -f(\theta_0)$ has the opposite sign (the middle equality holds since rotating by 2π means going all the way around the Earth). By the IVT there is then a point between θ_0 and its antipode where f vanishes.