Math 100 – SOLUTIONS TO WORKSHEET 2
LIMIT LAWS

1. Existence of limits and blowup

(1) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.

Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2-x^2) = 2 - 1^2 = 1$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\sqrt{x} = \sqrt{1} = 1$ so

$$\lim_{x\to 1} f(x) = 1\,.$$

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

Solution: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4-x^2) = 4 - 1^2 = 3$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\sqrt{x} = \sqrt{1} = 1$ so the limit does not exist (but the one-sided limits do).

(2) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

Solution: $f(x) = \frac{x-3}{(x-3)(x+4)} = \frac{1}{x+4}$ so $\lim_{x\to 3} f(x) = \frac{1}{3+4} = \boxed{\frac{1}{7}}$.

(b) What about $\lim_{x\to 2} f(x)$?

Solution: The limit does not exist: if x is very close to 2 then $x-2$ is very small and $\frac{1}{x-2}$ is very large. That said, when $x > 2$ we have $\frac{1}{x-2} > 0$ and when $x < 2$ we have $\frac{1}{x-2} < 0$ so (in the extended sense)

$$\lim_{x\to 2^+} \frac{1}{x-2} = +\infty$$
$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty\,.$$

More on this in the next lecture.

2. Limit Laws

Fact. *Limits respect arithmetic operations and standard functions (e^x, sin, cos, log, ...)* **as long as everything is well-defined.**

(beware especially of division by zero)

(3) Evaluate using the limit laws:

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1} =$

Solution: The expression is well-behaved at $x = 2$ so $\lim_{x\to 2} \frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2 - 1} = \frac{3}{15} = \frac{1}{5}$.

(b) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} =$

Solution: $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{e^x(x-1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{e^x}{x+2} = \frac{e^1}{1+2} = \frac{e}{3}$.

(4) Evaluate using the identity $\sqrt{a} - \sqrt{b} = \left(\sqrt{a} - \sqrt{b}\right) \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$:

Date: 14/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$.
Solution: Both numerator and denominator vanish at $x=0$ so we need to deal with the cancellation. Multiplying and dividing by $\sqrt{4+x}+2$ we have

$$\begin{aligned}\frac{\sqrt{4+x}-2}{x} &= \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} \\ &= \frac{(4+x)-4}{x\left(\sqrt{4+x}+2\right)} = \frac{x}{x\left(\sqrt{4+x}+2\right)} \\ \frac{1}{\sqrt{4+x}+2} &\xrightarrow[x\to 0]{} \frac{1}{\sqrt{4}+2} = \frac{1}{4}.\end{aligned}$$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$.
Solution: We have

$$\begin{aligned}\frac{\sqrt{1+x}-\sqrt{1+x^2}}{x^2} &= \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x^2} \cdot \frac{\sqrt{1+x}+\sqrt{1+x^2}}{\sqrt{1+x}+\sqrt{1+x^2}} \\ &= \frac{(1+x)-(1+x^2)}{x^2\left(\sqrt{1+x}+\sqrt{1+x^2}\right)} \\ &= \frac{x-x^2}{x^2\left(\sqrt{1+x}+\sqrt{1+x^2}\right)} \\ &= \frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}} \cdot \frac{1}{x}.\end{aligned}$$

Now as $x \to 0$ we have $\frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}} \to \frac{1}{2}$ while $\frac{1}{x}$ blows up so the whole expression blows up and the limit does not exist.

(5) Evaluate using the Sandwich/Squeeze Theorem

(a) $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$.
Solution: Since $-1 \le \sin\theta \le 1$ for all θ while $x^2 \ge 0$ we have for all x that

$$-x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2 .$$

Now $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}(-x^2) = 0$, so by the sandwich theorem $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$ too.

(b) (Final, 2014) Suppose that $8x \le f(x) \le x^2+16$ for all $x \ge 0$. Find $\lim_{x\to 4} f(x)$.
Solution: We have $\lim_{x\to 4} 8x = 32$ and $\lim_{x\to 4} x^2+16 = 32$ so by the sandwich theorem $\lim_{x\to 4} f(x)$ exists and equals 32.