7. Computing derivatives (5/10/2021)

Goals.

(1) The product and quotient rules
(2) The tangent line

Last Time.

Defined derivative $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$\iff f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Uses: (1) compute derivatives by definition

(2) recognize limits as derivatives

Calculating derivatives using rules:

$(af+bg)' = af' + bg'$ ("linearity")

$(x^a)' = a x^{a-1}$

Line tangent to graph $y=f(x)$ at $(a, f(a))$ has slope $f'(a)$, equation $y = f'(a)(x-a) + f(a)$

Math 100 – WORKSHEET 7
DIFFERENTIATION RULES

1. The product and quotient rules

(1) Differentiate

(a) $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

$f'(x) = 6\pi x^{\pi-1} + 2e x^{e-1} - \frac{7}{2} x^{5/2}$

(using linearity, rule for power laws)

(b) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)

$\frac{dg}{dx} = \frac{d(x^2)}{dx} \cdot e^x + x^2 \frac{d(e^x)}{dx} = 2x e^x + x^2 e^x$

$= (x^2 + 2x)e^x.$

$\boxed{\text{Warning: } g' \neq \frac{d(x^2)}{dx} \cdot \frac{d(e^x)}{dx}}$

Date: 5/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Why ~~is~~ are the rules true?

Derivative as linear approximation:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}.$$

$$\Leftrightarrow \lim_{x \to a} \left[\frac{f(x) - f(a)}{x-a} - f'(a) \right] = 0$$

$$\Leftrightarrow \lim_{x \to a} \frac{f(x) - [f(a) + f'(a)(x-a)]}{x-a} = 0$$

i.e. $f(x) = f(a) + f'(a)(x-a) + \overbrace{\text{remainder smaller than } (x-a)}^{R(x)}$

small: $\dfrac{R(x)}{x-a} \xrightarrow[x \to a]{} 0$.

Later: linear approximation

say: $f(x) = f(a) + f'(a) \cdot (x-a) + \text{small}$

$g(x) = g(a) + g'(a) \cdot (x-a) + \text{small}$

Then $(f+g)(x) = (f+g)(a) + (f'(a) + g'(a))(x-a) + \text{small}$

$(f \cdot g)(x) = (f \cdot g)(a) + f(a) g'(a)(x-a) + g(a) f'(a)(x-a) + f'(a) g'(a)(x-a)^2 + \text{small}$

$= (f \cdot g)(a) + (f(a) g'(a) + f'(a) g(a))(x-a) + \text{small}$

(c) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$

$$h'(x) = \frac{(x^2+3)' \cdot (2x-1) - (x^2+3)(2x-1)'}{(2x-1)^2} = \frac{2x(2x-1) - (x^2+3)(2)}{(2x-1)^2} =$$

$$= \frac{2x^2-2x-6}{(2x-1)^2}$$

(d) $\frac{x^2+A}{\sqrt{x}}$

$$\frac{x^2+A}{\sqrt{x}} = x^{3/2} + Ax^{-1/2} \quad \text{so} \quad \frac{d}{dx}\left(\frac{x^2+A}{\sqrt{x}}\right) = \frac{3}{2}x^{1/2} - \frac{A}{2} \cdot x^{-3/2}$$

Or : $\frac{d}{dx}\left(\frac{x^2+A}{\sqrt{x}}\right) = \frac{2x \cdot \sqrt{x} - (x^2+A) \cdot \frac{1}{2\sqrt{x}}}{x} = 2\sqrt{x} - \frac{1}{2}\sqrt{x} - \frac{A}{2\sqrt{x} \cdot x}$

$$= \frac{3}{2}\sqrt{x} - \frac{A}{2}x^{-3/2}.$$

(2) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.

Story:
(1) find $f'(x)$
(2) plug $x=4$ in
(3) equation for A.

$$f'(x) = \frac{\sqrt{x}+A}{(\sqrt{x}+A)^2} - \frac{x \cdot \frac{1}{2\sqrt{x}}}{(\sqrt{x}+A)^2} =$$

$\leftarrow (\sqrt{x}+A)'$

$$= \frac{\frac{1}{2}\sqrt{x}+A}{(\sqrt{x}+A)^2}$$

$\leftarrow$ computed f' without a value for A

So $\frac{3}{16} = f'(4) = \frac{\frac{1}{2}\cdot 2 + A}{(2+A)^2}$

So $\frac{3}{16}(2+A)^2 = 1+A$ or $\frac{3}{16}A^2 - \frac{1}{4}A - \frac{1}{4} = 0$

(3) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$. Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 10$

$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \frac{2}{2^2} = \frac{1}{2}$

2. The tangent line

(1) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

$f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$, so the tangent line is

$$y = \frac{1}{4}(x-4) + 2$$

(1) find slope
(2) create line through pt with right slope

(2) Let $f(x) = \frac{g(x)}{x}$, where $g(x)$ is differentiable at $x = 1$. The line $y = 2x - 1$ is tangent to the graph $y = f(x)$ at $x = 1$. Find $g(1)$ and $g'(1)$.

line has slope 2, point of tangency is $(1, 2\cdot 1 - 1) = (1, 1)$

so $f'(1) = 2$, $f(1) = 1$

But $f(1) = \frac{g(1)}{1}$ so $\boxed{g(1) = 1}$

And $f'(x) = \frac{g'(x)\cdot x - g(x)}{x^2}$ so $2 = f'(1) = \frac{g'(1)\cdot 1 - g(1)}{1}$

so $\boxed{g'(1) = 2 + g(1) = 3}$

(3) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

need $f(1) = 6$, $f'(1) = 4$. now check

(4) Find the lines of slope 3 tangent the curve $y = x^3 + 4x^2 - 8x + 3$.

If line is tangent at $x = a$, have $f'(a) = 3$

this is an equation for a:

$$\boxed{3a^2 + 8a - 8 = 3}$$

(Key step: gave the point of tangency name a)

(5) The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

don't know where the line is tangent to the curve.

Say line is tangent at $x = a$

then $\left(\frac{dy}{dx} = 3x^2 + 2\right)$ $\quad 3a^2 + 2 = 5$

slope of curve at a

slope of line

$\Rightarrow a = 1$ or $a = -1$

: